THE UNIVERSITY OF ALBERTA

AXIALLY SYMMETRIC GRAVITATIONAL FIELDS IN GENERAL RELATIVITY

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ABSTRACT

The object of this thesis is to survey and collate the extensive literature relating to axially symmetric vacuum solutions of Einstein's gravitational field equations, and to develop a number of new results.

"Space-time" considerations have imposed fairly severe restrictions on the range of topics covered. For instance, the large amount of work on approximate solutions, on axially symmetric gravitational waves and on solutions of combined Einstein-Maxwell field equations has been left entirely out of account. Within these limitations however, the author has aimed to be reasonably comprehensive.

We itemize briefly the main original contributions in the thesis.

§ 2.5. The theorem of page 32 is stated here for the first time. While a result of this type is implicit in the work of several earlier authors, it has not previously been given a precise and general formulation.

§ 3.3, page 50 and Appendix IV, page 118, the fields of oblate and prolate spheroids given here are based on the original calculations of the author.

§§ 3.4, 3.5, 3.6. Field of a set of collinear particles.

§ 4.3. Isolation of a new class of axially symmetric stationary fields.

Chapter V. Derivation of an explicit solution of Bondi's problem.

A number of other new results are scattered through the thesis. Where these involve novelty of substance rather than merely of presentation they have been explicitly indicated as such.

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CHAPTER I

HISTORICAL REVIEW

This chapter is intended to review the work done by authors on vacuum gravitational fields which possess axial symmetry in general relativity.

Such a review of work is appropriate at the present time because the renaissance of interest in general relativity is yielding a flood of new solutions, by using recently developed techniques (e.g. algebraically special fields). At the same time, a considerable body of solutions were discovered and rediscovered by many authors in the 1920's and 1930's, and appeared in numerous, but not easily accessible journals. One of the purposes which this thesis will serve, will be to collect some of these results under one roof.

In the extraordinary burst of activity in Germany and other countries, which followed publication of Einstein's gravitational theory, the theory of static axially symmetric fields was one of the earliest and most important achievements. Einstein's definitive formulation of his theory of gravitation appeared in the "Sitzungsberichte der Preussischen Akademie der Wissenschaften" in late 1915 [1]. Just two or three months later the same journal carried the first exact non-trivial solution of the fields equations - Karl Schwarzschild's spherical symmetric solution representing the static exterior field of a massive sphere or a particle [2].

In a classic paper published in 1917, Weyl [3] showed that by introducing suitable coordinates - "canonical cylindrical coordinates"

for static axially symmetric fields

r, z, φ - the vacuum field equations could be reduced to the determination of two functions $\lambda(r, z)$, $\nu(r, z)$, the first satisfying Laplace's equation

$$\frac{\partial^2\lambda}{\partial r^2} + \frac{1}{r}\frac{\partial\lambda}{\partial r} + \frac{\partial^2\lambda}{\partial z^2} = 0$$

and the second a Poisson equation.

Levi-Civita [4] working at almost the same time, further simplified the theory of static, axially symmetric gravitational fields. He showed that ν is obtainable from λ by a straight quadrature.

In this way, a large class of exact solutions of the field equations was made available. Some idea of Weyl's mathematical power and of the difficulties of the problem can be gained from the fact that no significant widening of this class has been achieved in the almost half-century since the appearance of Weyl's paper. What little is known about exact non-static solutions has often been won by using some simple variant of the Weyl-Levi-Civita formalism. For instance, Einstein and Rosen [5] and later workers [6] applied the complex transformation $z \to it$, $t \to iz$ to Weyl's line-element to discuss cylindrical gravitational waves. In this category belongs too, the essay of Bondi [7] on the motion of two masses $+m$, $-m$ discussed in detail in Chapter V.

In 1922, Weyl's pupil R. Bach [8] used his techniques to derive a number of explicit axially symmetric solutions, in particular the field of a circular ring and the static field of two particles. Already in 1917, Weyl had indicated that the Schwarzschild solution is obtained if one formally puts λ proportional to the Newtonian potential of a uniform rod.

To obtain his 2-particle solution, Bach chose λ equal to the potential of two such rods placed along the z-axis.

In a supplement to Bach's paper, Weyl [9] pointed out a pit-fall into which duly fell a succession of later authors [10-13] who, unaware of this work, independently rediscovered Bach's formal result or variants of it. It is possible that the singularity of the cylindrical co-ordinates on the axis of symmetry $r = 0$ may mask a true singularity of the space-time manifold on this axis. By paying due attention to this point, Bach and Weyl were able to show that there is in fact a singularity of the manifold along the line joining the two particles. This singularity is nowadays interpreted as a strut which holds the two particles apart, and keeps them at rest.

In his paper [9], Weyl established the general result that Einstein's field equations admit no axially symmetric solution representing two static bodies completely separated by a surface z = constant, which is non-singular everywhere outside the two bodies. This was one of the first anticipatory clues to the most remarkable and significant feature of Einstein's theory, a feature which still distinguishes it from all other present-day field theories. The non-linearity of the field equations means that if φ_1 , φ_2 are solutions each representing a static singularity, it is impossible to superimpose these to get a static 2-body solution $\varphi_1 + \varphi_2$. A solution having two singularities exists only under special circumstances - provided the two singularities move in the right way! The equations of motion are in fact built into the field equations. There is, for instance, no need to assume as a separate postulate that test-particles follow geodesics in

space-time, as Einstein did in 1915 - this is a consequence of the field equations [16].

However, it was left to Einstein in 1927 [17] to extract the full physical implications of Weyl's result and with his collaborators, to develop the detailed theory of gravitational motion [18]. In 1922, Weyl merely stated [9, p.145]

> "Die physikalische Bedeutung dieses Resultats wird man nicht übertreiben dürfen; für die Lösung des wirklichen Zweikörper - Problems, die Bestimmung der Bewegung zweier sich anziehender schwerer Massen, ist damit nichts gewonnen. Aber es ist doch ein physikalisch sinnvoller exakter Ausdruck gewonnen für die Kraft mit der sich zwei Massenpunkte nach der Einsteinschen Theorie anziehen".

Perhaps because of post-war communication difficulties, the work of Bach and Weyl, and even the earlier work of Weyl and Levi-Civita, remained for some time largely unknown outside the countries of their authors. In particular, the work of most English writers seems to have been completely independent. W. Wilson [19] obtained in 1920 the external field of an infinite rod or cylinder. In 1924, H. E. J. Curzon [12] formally derived a static solution representing two "particles" in which λ was given by

$$\lambda = -\left[\frac{m_1}{\sqrt{r^2+(z-a_1)^2}} + \frac{m_2}{\sqrt{r^2+(z-a_2)^2}}\right] .$$

This agrees with Bach's solution only asymptotically at large distances, and unlike it, does not reduce to Schwarzschild's solution when one of the masses vanishes. Curzon also failed to recognize that his solution becomes singular on the axis of symmetry between the particles. In 1936, in a paper entitled

"Two-centres solution of the gravitational field equations and the need for a reformed theory of matter" L. Silberstein [13] rediscovered Curzon's result, again overlooking the axial singularity (c.f. the reply by Einstein and Rosen [14]). In the meantime, Bach's formal result had been rediscovered by Chazy [10] in France, and Palatini [11] in Italy. A rather obscure paper on the same subject by E. Trefftz [15] appeared almost simultaneously with Bach's in 1922.

Recently, Marder [20] has given an interesting new interpretation to Curzon's solution: it may be regarded as the exterior field of a body encased in a second hollow body. Bach's solution can of course be similarly interpreted [see Chapter III].

The extension of Curzon's solution to the static field of n "collinear Curzon particles" was first explicitly carried out quite recently by Hoffmann [21]. It was shown by Bergmann [22] that such a solution necessarily involves extraneous singularities if the masses are all assumed positive. This corresponds to the fact that there is no Newtonian equilibrium configuration for n collinear positive masses. However, Hoffmann pointed out that static solutions without struts exist if the particles are allowed to have masses of either sign. Similar remarks have been made by Synge [23]. Even in this case, it seems that no strut-free static solution exists if n is even. The corresponding generalization of Bach's solution to n collinear "Schwarzschild particles" is given for the first time in this thesis (Chapter III).

Closely related to this is a problem first considered by Bondi [7] who has attempted to examine the dynamics of a pair of equal and opposite

masses. Simple arguments based on the principle of equivalence lead to the expectation that the two masses if started with the same velocity will move as a rigid, uniformly accelerated system* - the acceleration being directed from the negative to the positive mass. He obtained a vacuum line-element whose domain of validity does not cover the whole space-time, but only covers a sector, one quarter of the whole space-time bounded by light-like asymptotes. He was able to prove the global existence of another type of solution involving four particles. His solution possesses reflexional symmetry and represents two identical Bondi dipoles moving in radially opposite directions. We discuss this problem in detail in Chapter V.

Very recently Bonnor and Swaminarayan [24] have presented exact global solutions of Einstein's equations representing four Curzon-type particles in motion. One of their metrics is an explicit global solution of Bondi's 4-body problem for the case of four Curzon particles (c.f. our simpler solution for four Schwarzschild particles). They have also presented solutions corresponding to the case where all four masses are positive. In this case, struts along the z-axis are necessary to maintain the motion.

On the basis of Weyl-Levi-Civita formalism, Chou [25] investigated the static axially symmetric fields of prolate and oblate spheroidal homeoids.

Erez and Rosen [26] have given the static field of a particle possessing a mass quadruple moment.

In 1960, Misra [27] using Weyl's technique, determined the external gravitational field of an oblate spheroid.

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* - called a Bondi dipole in this thesis.

Unaware of Misra's work, the present author carried out calculations for oblate and prolate spheroids. These calculations are presented in Appendix IV. The final result agrees with Misra's work in the case of oblate spheroids.

We pass from static solutions to stationary solutions with axial symmetry, corresponding physically to the field of an axisymmetric body in steady rotation. Approximation methods for dealing with this case have received considerable attention. The first approximate solution was given by Lense and Thirring [28]. Bach [29], Andress [30], Akeley [31], Das, Florides and Synge [32] and Florides and Synge [33] have also used the approximation method. However, this lies outside the scope of this thesis.

The field equations for stationary axially symmetric vacuum fields emerge as a complicated set of non-linear partial differential equations. As regards exact solutions, it has not yet proved feasible to obtain more than a few particular results. Lewis [34] found a solution with cylindrical symmetry which he interpreted as the external field of an infinite rotating cylinder.

A corresponding interior solution was derived by Van Stockum [35]. Some other particular solutions have been given by Papapetrou [36], Tiwari and Misra [37], but they do not seem amenable to a realistic physical interpretation. In general, it is fair to say that the approach of Andress, Lewis and their successors has not to date yielded anything of real physical value.

Very recently, Roy Kerr [38] investigating the class of empty space-

times which are characterized by the existence of a geodesic, shear-free null congruence, obtained a remarkable particular solution which is stationary, possesses axial symmetry and contains the Schwarzschild solution as a special case. He has suggested that his solution represents the field of a spinning particle. Still more recently, Newman and Janis [39] have put forward arguments supporting their interpretation of Kerr's solution as the field of a uniform circular ring spinning about its axis.

CHAPTER II

AXIALLY SYMMETRIC VACUUM FIELDS

INDEPENDENT OF TIME: GENERAL THEORY

§2.1 Einstein's theory of gravitation.

In its mathematical aspects, general relativity is the study of a four-dimensional Riemannian space ("space-time"). In terms of an admissible co-ordinate system, the squared metric interval between two neighbouring points or "events" x^i and $x^i + dx^i$ is given by the quadratic differential form

$$\Phi = g_{ij}(x)\, dx^i\, dx^j$$

with repeated latin indices summed from 1 to 4. The signature of the metric is assumed to be +2 , i.e. g_{ij} is locally reducible at any event to the diagonal (1, 1, 1, -1). The separation dx^i of two events is called space-like, time-like or null according as Φ is positive, negative or zero.

The contravariant metric tensor g^{ij} is defined as

$$g^{ij} = \frac{G(i,j)}{g} \tag{2.1}$$

where $G(i,j)$ is the cofactor of g_{ij} in the determinant $g = |g_{ij}| \neq 0$.

The two kinds of Christoffel symbols formed from the fundamental tensor g_{ij} are defined as

(2.2) $$\Gamma_{ij,k} = \tfrac{1}{2}(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}) \quad , \quad \text{where} \quad \partial_j = \frac{\partial}{\partial x^j}$$

and

$$\Gamma^h_{ij} = g^{hk}\, \Gamma_{ij,k} \quad .$$

From these symbols we define a fourth order tensor

(2.3) $$R^h_{.ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^h_{jm}\, \Gamma^m_{ik} - \Gamma^h_{km}\, \Gamma^m_{ij}$$

called the "Riemann-Christoffel tensor".

By contraction of this tensor, we have

(2.4) $$R_{ij} = R^h_{\cdot ihj} = \partial_h \Gamma^h_{ij} - \partial_j \Gamma^h_{ih} + \Gamma^h_{hm}\, \Gamma^m_{ij} - \Gamma^h_{mj}\, \Gamma^m_{ih} \quad .$$

This symmetric tensor R_{ij}, which plays a fundamental role in relativistic gravitational theory, is known as the Ricci tensor.

Contracting the Ricci tensor we obtain the invariant

(2.5) $$R = R^i_i = g^{ij} R_{ij}$$

called the "curvature invariant".

The Einstein tensor is defined by

(2.6) $$G_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R \quad .$$

The conditions

(2.7) $$R^{h}_{\cdot ijk} = 0$$

are necessary and sufficient for flatness of the space-time manifold, i.e. reducibility of the g_{ij} everywhere to diag (1, 1, 1, -1).

For the physical interpretation of space-time, we say that a physical "observer" or a "frame of reference" is identified with an orthogonal triad of space-like vectors propagated by Fermi transport [23] along a time-like curve - the "world line" or the "history" of the observer. The observer correlates arc-length along his world-line with time registered by his clocks and the orthogonal triad with a set of fixed axes in his laboratory to which he refers incoming and outgoing light-signals, test-particles, etc. If the separation of two neghbouring events is time-like, then both "happen" to some observer (i.e. his world-line encounters both the events) and the metric interval between them is his measure of the time-lapse between their occurrence. If their separation is space-like, then observers can be found for whom they are simultaneous, and the metric interval is the spatial distance (as measured by such observers) between their places of occurrence.

The dynamics of particles and photons in a gravitational field is governed by the geodesic hypothesis, actually a consequence of the field equations (2.10): the world-lines of spinless particles subject only to gravitational influence are time-like geodesics. The history of a light flash is a null geodesic.

To complete the theory, it is necessary to specify how the geometrical structure of space-time is determined by the distribution of

matter.

The <u>field-equations</u> of general relativity (analogous to Poisson's equation $\nabla^2 V = 4\pi\rho$ in Newton's theory) are

$$G_{ij} = k\, T_{ij} \quad , \tag{2.8}$$

where k is a contant, and T_{ij} is the energy tensor of the material generating the gravitational field. For a perfect fluid with density μ, pressur P, and the 4-velocity u_i, T_{ij} has the form

$$T_{ij} = (\mu + P)\, u_i u_j + P g_{ij} \quad . \tag{2.9}$$

We shall for the most part confine our attention to solutions valid in regions free from matter. This means we have to solve the vacuum field equations under certain suitable initial/boundary/junction conditions. One of the conditions is normally taken to be asymptotic flatness at spatial infinity.

The vacuum field equations are

$$G_{ij} = 0 \tag{2.10}$$

or, equivalently,

$$R_{ij} = 0 \; . \tag{2.11}$$

This is a set of ten partial differential equations for the ten g_{ij}, quasi-linear in the second derivatives and quadratic in the first derivatives.

The existence of four differential identities

(2.12) $$\partial_j G^{ij} + G^{mj}\Gamma^i_{mj} + G^{im}\Gamma^j_{mj} \equiv 0$$

between the equations (2.10) means that four arbitrary functions will appear in the solution for g_{ij}. This is precisely the amount of arbitrariness to be expected on account of the arbitrariness of the coordinate system.

To sum up: <u>In general relativity, a gravitational field in empty space is characterized by a normal hyperbolic Riemannian space of four dimensions with vanishing Ricci tensor</u>.

§2.2 Symmetries and groups of motions.

(For a detailed discussion of the material of this section, the reader is advised to see [40], [41], [42] in the Bibliography.)

Suppose that for each point $P(x^\mu)$ of space-time, there is a corresponding point $\bar{P}(\bar{x}^\mu)$ which is physically equivalent to it in the sense that all intrinsic properties of the world as viewed by an observer at P are indistinguishable from those as viewed by an observer at $\bar{P}$. Then the geometrical structure of space-time is unchanged by going from P to $\bar{P}$, i.e. the point transformation

(2.13) $$x^\mu \rightarrow \bar{x}^\mu \equiv \psi^\mu(x)$$

is a <u>rigid motion</u> of space-time:

i.e. $$\Phi(\bar{P}, \bar{Q}) = \Phi(P, Q)$$

for two neighbouring points $P(x^i)$, $Q(x^i + dx^i)$, or

$$g_{ij}(\bar{x})\, d\bar{x}^i\, d\bar{x}^j = g_{ij}(x)\, dx^i\, dx^j \quad . \tag{2.14}$$

Let $x^\mu = \varphi^\mu(\bar{x})$ denote the inverse of the point transformation (2.13), and introduce the new co-ordinates

$$x^{\mu'} = \varphi^\mu(x) \quad . \tag{2.15}$$

Then $\bar{P}$ has the same numerical co-ordinate values in the primed system $(\bar{x}^{\mu'} = x^\mu)$ as P has in the original system of co-ordinates (i.e. co-ordinates "dragged along" with the motion).

Since Φ is invariant under co-ordinate transformations,

$$\begin{aligned} g_{ij}(\bar{x})\, d\bar{x}^i\, d\bar{x}^j &= g'_{ij}(\bar{x}')\, d\bar{x}^{i'}\, d\bar{x}^{j'} \\ &= g'_{ij}(x)\, dx^i\, dx^j \quad . \end{aligned} \tag{2.16}$$

Here, $g'_{ij}(x)$ means: compute $g'_{ij}(x')$ from (2.15) and the tensor law of transformation

$$g'_{ij}(x') = g_{ab}(x) \frac{\partial x^a}{\partial x^{i'}} \frac{\partial x^b}{\partial x^{j'}} \tag{2.17}$$

and then evaluate the left hand side at $x'^\mu = x^\mu$. Comparing (2.14) and (2.16), we have as the condition for a rigid motion

$$g'_{ij}(x) = g_{ij}(x) \tag{2.18}$$

or

$$[g'_{ij}(x')]_{x'=x} = g_{ij}(x) \quad . \tag{2.19}$$

Hence space-time admits a rigid motion if and only if the following condition is satisfied:

<u>There exists a co-ordinate transformation $x^{\mu'} = \varphi^{\mu}(x)$ such that $g'_{ij}(x')$ is the same function of $x^{p'}$ as $g_{ij}(x)$ is of x^{p} .</u>

If we now assume that $\bar{P}$ can be reached from P by a path consisting of points equivalent to P , then the condition $g_{ij}(\bar{x})\, d\bar{x}^i\, d\bar{x}^j = g_{ij}(x)\, dx^i\, dx^j$ will be satisfied for the point-transformation

$$\bar{x}^{\mu} = \psi^{\mu}(x, \sigma) \tag{2.20}$$

containing a continuous parameter σ , for some value of which (say $\sigma = 0$), (2.20) becomes the identity transformation $\bar{x}^{\mu} = x^{\mu}$.

The set of transformations (2.20) form a <u>group</u>, for it is clear from (2.14) that if P_1 is equivalent to $\bar{P}$ and $\bar{P}$ is equivalent to P , then P_1 is equivalent to P . For the infinitesimal transformation of the group

$$\bar{x}^{\mu} = x^{\mu} + \xi^{\mu}(x)\, \delta\sigma \quad , \tag{2.21}$$

where

$$\xi^{\mu}(x) = \left[\frac{\partial\psi^{\mu}(x,\sigma)}{\partial\sigma} \right]_{\sigma=0}$$

(2.14) yields

$$g_{ij}(x+\xi\delta\sigma)(dx^i+d\xi^i\delta\sigma)(dx^j+d\xi^j\delta\sigma) \tag{2.22}$$

$$= g_{ij}(x)\, dx^i\, dx^j$$

i.e.

$$\xi^k \partial_k g_{ij} + g_{ik}\partial_j \xi^k + g_{jk}\partial_i \xi^k = 0 \quad , \tag{2.23}$$

which can be written in the form

$$\xi_{i/j} + \xi_{j/i} = 0 \quad . \tag{2.24}$$

These are Killing's equations for the infinitesimal generator ξ_i of a continuous group of rigid motions. As is clear from (2.23), they also impose restrictions on the geometric structure of space-time.

Let the four functions $\varphi^j(x)$ be independent solutions of

$$\xi^i(x)\,\frac{\partial\varphi^j}{\partial x^i} = 0 \quad (j = 1, 2, 3)$$

$$= 1 \quad (j = 4)$$

and make the co-ordinate transformation $x^{j'} = \varphi^j(x)$. In the new co-ordinate system we have

$$\xi^{j'} = \delta^j_4 \quad .$$

Hence (see [41]) :

Co-ordinates can always be found which make the components of the infinitesimal generator of a one-parameter group reduce to

$$\xi^1 = \xi^2 = \xi^3 = 0 , \quad \xi^4 = 1 .$$

In these co-ordinates, the finite equations of the group are [c.f. (2.21)]

(2.25) $$\bar{x}^{\mu} = x^{\mu} \ (\mu = 1, 2, 3), \ \bar{x}^4 = x^4 + \sigma ,$$

and the Killing equations (2.23) reduce to

(2.26) $$\partial_4 g_{ij} = 0 ,$$

a result immediately obvious from (2.14) and (2.25).

§2.3 Stationary and static space-time with axial symmtery.

A space-time manifold will be called stationary if it admits a one-parameter group of time-like rigid motions, that is, there exists a vector ξ^i satisfying (2.23) and $g_{ij}\xi^i\xi^j < 0$. According to the last-result of §2.2 , this means that there exists a co-ordinate system $x^1, x^2, x^3, x^4 = t$, with t a time-like co-ordinate, such that the metric tensor $g_{ij}(x^1, x^2, x^3)$ is independent of t . This co-ordinate system is fixed to within an arbitrary "spatial transformations"

$$x^{\mu'} = \varphi^{\mu}(x^1, x^2, x^3)$$

where $\mu = 1, 2, 3$ and to arbitrary time translations, $t' = t + \text{constant}$, since these preserve the relations

$$\xi^i = (0, 0, 0, 1) \ ,$$

$$\bar{x}^\mu = x^\mu (\mu = 1, 2, 3), \quad \bar{x}^4 = x^4 + \sigma$$

$$\partial_4 g_{ij} = 0 \ .$$

A stationary space-time is called <u>axially symmetric</u> if

(i) <u>there exists one-parameter group of space-like rigid rotations whose generating vector η_i is orthogonal to ξ^i :</u>

(2.27) $$g_{ij}\eta^i\eta^j > 0 \ , \quad g_{ij}\xi^i\eta^j = 0$$

$$\eta_{i/k} + \eta_{k/i} = 0 \ .$$

The orthogonality of ξ^i , η^i implies that η^i defines a rigid motion in each hypersurface t = constant. Hence by a spatial transformation we can introduce co-ordinates $x^1, x^2, x^3 = \varphi$ such that $g_{ij}(x^1, x^2)$ is independent of φ .

(ii) <u>Space-time geometry is invariant under the simultaneous reversal of φ and t :</u>

(2.28) $$\varphi \to \bar{\varphi} = - \varphi \ , \quad t \to \bar{t} = - t \ .$$

According to (2.14) this gives

$$(2.29)\qquad \bar{g}_{AB}dx^A dx^B - 2\bar{g}_{A3}dx^A d\varphi - 2\bar{g}_{A4}dx^A dt + \bar{g}_{33}d\varphi^2 + 2\bar{g}_{34}d\varphi dt + \bar{g}_{44}dt^2$$

$$= g_{AB}dx^A dx^B + 2g_{A3}dx^A d\varphi + 2g_{A4}dx^A dt + g_{33}d\varphi^2 + 2g_{34}d\varphi dt + g_{44}dt^2$$

where $\bar{g}_{ij} = g_{ij}\ (-\varphi, -t)$ and $(A,B = 1,2)$. But since g_{ij} are independent of φ and t , we have $\bar{g}_{ij} = g_{ij}$, and we infer from (2.29) that

$$(2.30)\qquad g_{14} = g_{24} = g_{13} = g_{23} = 0 \quad .$$

We briefly indicate the intuitive motivation for the above definition. A stationary axially symmetric gravitational field is the kind of field produced by an axially symmetric distribution of matter rotating steadily about its axis of symmetry. An electromagnetic analogue would be the field of an axially symmetric charge distribution in steady rotation. Imagine a compass inserted to show the direction of the magnetic field, and a film taken of the phenomenon. The film, if run backwards $(t \to -t)$ will <u>not</u> give an accurate picture of the field produced by the same distribution rotating in the opposite sense, since the magnetic field ought to change direction but does not. However, if we somehow reverse the sense of rotation $(\varphi \to -\varphi)$ while running the film backwards, we return to an accurate picture of the original field. Condition (ii), thus gives expression to the hypothesis that a stationary axially symmetric gravitational field is indifferent to time inversion if we at the same time reverse the sense of rotation of the masses producing the field. It is well to add that while this (or some equivalent) hypothesis has usually been made in

discussing such fields, the case for it is not overwhelmingly strong.

Summing up, in a stationary axially symmetric space-time, it is possible to introduce co-ordinates $x^1, x^2, x^3 = \varphi, x^4 = t$ in which the metric reduces to

$$\Phi = g_{AB}dx^A dx^B + 2g_{34}d\varphi dt + g_{33}d\varphi^2 + g_{44}dt^2 \tag{2.31}$$

where $(A, B = 1, 2)$,

$$g_{ij} = g_{ij}(x^1, x^2),\ g_{44} < 0\ ,\ g_{33} \geq 0\ ,\ \ g_{AB}dx^A dx^B \geq 0\ .$$

A further simplification is possible. We recall that any two-dimensional line-element

$$g_{AB}(x^1, x^2)\ dx^A\ dx^B$$

is reducible to "isothermal form" $[A(x^{1'}, x^{2'})]^2[(dx^{1'})^2+(dx^{2'})]^2$ by a co-ordinate transformation. This is immediately seen. Consider the two families of (complex) null-curves

$$u(x^1,x^2) = \text{constant},\quad v(x^1,x^2) = \text{constant}\ ,$$

whose infinitesimal tangent vectors satisfy

$$g_{AB}dx^A\ dx^B = 0\ ;$$

v is the complex conjugate of u. In terms of the null co-ordinates u, v the metric is

$$g_{AB}dx^A\ dx^B = [A(u,v)]^2\ du\ dv$$

Defining the real co-ordinates

$$x^{1'} = \tfrac{1}{2}(u+v), \quad x^{2'} = \frac{1}{2i}(u-v)$$

the metric becomes

$$A^2[(dx^{1'})^2 + (dx^{2'})^2] \; .$$

Thus we have succeeded in reducing the metric tensor for a stationary axially symmetric field to a form in which the only non-vanishing off-diagonal element is g_{34} .

We shall now make a further specialization:

<u>Definition:</u> - A space-time is called STATIC if it is STATIONARY and invariant under time-inversion $\bar{t} = -t$.

Intuitively, a static field is produced by matter in a state of rest.

By an argument similar to that leading to (2.30), we find that the static condition is equivalent to

(2.32) $$g_{14} = g_{24} = g_{34} = 0 \quad .$$

This means, geometrically, that the t-lines are orthogonal to the isometric space-like hypersurfaces t = constant.

We have now shown that <u>in a static, axially symmetric space-time co-ordinates $x^1, x^2, x^3 = \varphi, x^4 = t$ can be found which reduce the metric to the form</u>

(2.33) $$\Phi = A^2[(dx^1)^2 + (dx^2)^2] + B^2(dx^3)^2 - C^2(dx^4)^2$$

where A, B, C are functions of x^1, x^2 .

§2.4 Weyl fields.

Following Weyl [3] and Levi-Civita [4], we now consider the solution of Einstein's field equations for static, axially symmetric vacuum fields - often called Weyl fields.

From (2.33)

$$\Phi = A^2[dx^1)^2 + (dx^2)^2] + B^2(dx^3)^2 - C^2(dx^4)^2$$

where A, B, C are functions of x^1, x^2 .

From the components of Ricci tensor corresponding to (2.33), which we have given in the Appendix I for convenience, it follows that

(2.34) $$R^3_3 + R^4_4 = \frac{R_{33}}{B^2} - \frac{R_{44}}{C^2} = \frac{1}{A^2BC}\left\{\frac{\partial^2(BC)}{(\partial x^1)^2} + \frac{\partial^2(BC)}{(\partial x^2)^2}\right\}$$

In a matter-free domain, the field equations are

(2.35) $$R_{ij} = 0 \; .$$

Hence

(2.36) $$R^3_3 + R^4_4 = \frac{1}{A^2BC}\left\{\frac{\partial^2(BC)}{(\partial x^1)^2} + \frac{\partial^2(BC)}{(\partial x^2)^2}\right\} = 0$$

or

(2.37) $$\frac{\partial^2(BC)}{(\partial x^1)^2} + \frac{\partial^2(BC)}{(\partial x^2)^2} = 0$$

i.e. BC is a <u>harmonic function</u> of (x^1, x^2) . Let

(2.38) $$BC = r(x^1, x^2)$$

and suppose z is the conjugate function of r , then

(2.39) $$r + iz = f(x^1 + ix^2)$$

where $f(x^1 + ix^2)$ is an analytic function in $x^1 + ix^2$.

In making the conformal transformation from (x^1, x^2) to (r, z) we have

(2.40) $$A^2[(dx^1)^2 + (dx^2)^2] = \alpha^2[dr^2 + dz^2]$$

where α is a function of (r, z) .

From (2.38)

(2.41) $$B = \frac{r}{C} .$$

Writing

(2.42) $$x^1 = r \, , \; x^2 = z \, , \; x^3 = \varphi \;\text{ and }\; x^4 = t$$

the metric form (2.33) can be written as

(2.43) $$\alpha^2(dr^2 + dz^2] + r^2C^2d\varphi^2 - C^2dt^2$$

where A and C are functions of (r, z) .

Hence we may state that in any static axially symmetric domain, to reduce the metric to the form (2.43), it is sufficient that $R_3^3 + R_4^4 = 0$.

In order to avoid cumbersome mathematical manipulations to determine the vacuum equations both Weyl [3] and Levi-Civita [4] assume that

$$A = e^{\nu-\lambda} \;,\; B = re^{-\lambda} \text{ and } C = e^{\lambda}$$

λ and ν are functions of (r, z) .

Hence (2.43) becomes

$$e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2 e^{-2\lambda} d\varphi^2 - e^{2\lambda} dt^2 \quad . \tag{2.44}$$

The expressions for the non-vanishing components of the Ricci tensor corresponding to (2.44) are

$$\begin{aligned} R_{11} &= \nu_{11} + \nu_{22} - \lambda_{11} - \lambda_{22} + 2\lambda_1^2 + \frac{\nu_1 - \lambda_1}{r} \\ R_{22} &= \nu_{11} + \nu_{22} - \lambda_{11} - \lambda_{22} + 2\lambda_2^2 - \frac{\nu_1 - \lambda_1}{r} - \frac{2\lambda_1}{r} \\ R_{12} &= 2\lambda_1 \lambda_2 - \frac{\nu_2}{r} \\ R_{33} &= - e^{-2\nu} r^2 [\lambda_{11} + \frac{1}{r}\lambda_1 + \lambda_{22}] \\ R_{44} &= - e^{-2\nu + 4\lambda} [\lambda_{11} + \frac{1}{r}\lambda_1 + \lambda_{22}] \end{aligned} \tag{2.45}$$

where $\lambda_1 = \frac{\partial\lambda}{\partial r}$, $\lambda_2 = \frac{\partial\lambda}{\partial z}$ and so on .

In a matter-free domain, we have

$$\tfrac{1}{2}(R_{11} + R_{22}) = \Delta\nu - (\Delta\lambda + \frac{\lambda_1}{r}) + \lambda_1^2 + \lambda_2^2 = 0$$

where

$$\Delta\nu = \frac{\partial^2\nu}{\partial r^2} + \frac{\partial^2\nu}{\partial z^2} \; ; \; \Delta\lambda = \frac{\partial^2\lambda}{\partial r^2} + \frac{\partial^2\lambda}{\partial z^2}$$

$$\tfrac{1}{2}(R_{11} - R_{22}) = \lambda_1^2 - \lambda_2^2 - \frac{\nu_1}{r} = 0$$

$$R_{12} = 2\lambda_1\lambda_2 - \frac{\nu_2}{r} = 0$$

$$R_{33} = -e^{-2\nu} r^2[\lambda_{11} + \frac{1}{r}\lambda_1 + \lambda_{22}] = 0$$

$$R_3^3 + R_4^4 = 0$$

i.e.

(2.46) $$\nabla^2\lambda \equiv \lambda_{11} + \frac{1}{r}\lambda_1 + \lambda_{22} = 0$$

(2.47) $$\nu_1 = r\{\lambda_1^2 - \lambda_2^2\}$$

(2.48) $$\nu_2 = 2r\lambda_1\lambda_2$$

(2.49) $$\Delta\nu + \lambda_1^2 + \lambda_2^2 = 0 .$$

Thus λ satisfies the Laplace's equation (2.46) in cylindrical co-ordinates (r, φ, z) in Euclidean 3-space and ν satisfies the Poisson equation.

Hence by the introduction of "canonical cylindrical co-ordinates (r, z, φ)", Weyl [3] and Levi-Civita [4] reduced the problem of obtaining any static,axially, symmetric vacuum field equations to the determination of two functions

$$\lambda(r, z) \quad \text{and} \quad \nu(r, z)$$

$\nu(r, z)$ is obtained from λ by a straight quadrature

$$(2.50) \qquad \nu = \int r \left\{ \left[\left(\frac{\partial\lambda}{\partial r}\right)^2 - \left(\frac{\partial\lambda}{\partial z}\right)^2 \right] dr + 2\frac{\partial\lambda}{\partial r} \cdot \frac{\partial\lambda}{\partial z} \; dz \right\}$$

§2.5 <u>Elementary flatness</u>.

We come now to a slightly subtle point which, as we have explained in Chapter I, was overlooked by many early authors. Weyl's canonical cylindrical co-ordinates become singular on the axis of symmetry $r = 0$. To discover whether this co-ordinate singularity conceals a true singularity of the manifold at a point $z = z_1$ of the axis, one can check whether the 2-space S formed by drawing all geodesics normal to the z-axis at z_1 has finite Gaussian curvature at z_1 .

We recall that the Gaussian curvature K of a 2-space at any point can be computed by surrounding the point by a small geodesic circle of circumference C and a geodesic radius ρ ; then [43]

$$(2.51) \qquad K = \lim_{\rho \to 0} \frac{2\pi\rho - C}{\frac{1}{3}\pi\rho^3} \; .$$

Thus, for the absence of singularities, it is necessary that

$$2\pi - \frac{C}{\rho} = O(\rho^2) \qquad \rho \to 0$$

for every geodesic circle.

Intuitively, (2.51) expresses the condition that the 2-space should possess a tangent plane at the point $\rho = 0$.

In particular, for a geodesic circle in S about $r = 0$, $z = z_1$, we have from (2.44)

$$\rho \sim re^{(\nu-\lambda)} \quad ; \quad C \sim 2\pi re^{-\lambda} \quad (r \to 0) \ .$$

<u>Hence the point $r = 0$, $z = z_1$ is a true singularity of the manifold (2.44) unless</u>

(2.52) $$\nu(0, z_1) = 0 \ .$$

Now a singularity of K on the z-axis implies a singularity of the Riemann tensor. This implies the violation of the vacuum equations $R_{ij} = 0$ on the z-axis, and therefore the presence of a material strut on the z-axis.

It was Weyl [9] who first pointed out the remarkable repercussions of elementary flatness condition on the existence of axially symmetric static 2-body solutions in general relativity. We shall discuss this in detail with respect to <u>particular solutions</u> in Chapter III. The general argument which follows is a simplification and generalization of those due to Weyl [9] and Bondi [7].

Let the metric

$$\Phi = e^{2(\nu-\lambda)}(dr^2+dz^2) + r^2 e^{-2\lambda} d\varphi^2 - e^{2\lambda} dt^2$$

represent the static, axially symmetric field of a set of discrete bodies $B, B_1, B_2, \ldots$. This means the vacuum field equations

(2.53) $$\nabla^2 \lambda = 0$$

(2.54) $$\frac{\partial \nu}{\partial r} = r\{(\frac{\partial \lambda}{\partial r})^2 - (\frac{\partial \lambda}{\partial z})^2\}$$

(2.55) $$\frac{\partial \nu}{\partial z} = 2r \frac{\partial \lambda}{\partial r} \cdot \frac{\partial \lambda}{\partial z}$$

are satisfied everywhere except within the shaded regions $B, B_1, B_2 \ldots$ in the Euclidean map of the cylindrical co-ordinates (r, z, φ) (see Fig. 1)

Let us suppose that all matter is contained within a finite distance from the origin. Then (like the Newtonian potential which it asymptotically approaches) λ behaves at spatial infinity according to

(2.60) $$\lambda(r, z) = O(\frac{1}{\rho}) \qquad (\rho \to \infty)$$

(2.61) $$(\frac{\partial \lambda}{\partial r}, \frac{\partial \lambda}{\partial z}) = O(\frac{1}{\rho^2})$$

where

$$\rho^2 = r^2 + z^2 .$$

From (2.55) we have ν = constant on segments of z-axis which are free of matter. Let us suppose the arbitrary constant in ν is adjusted to make

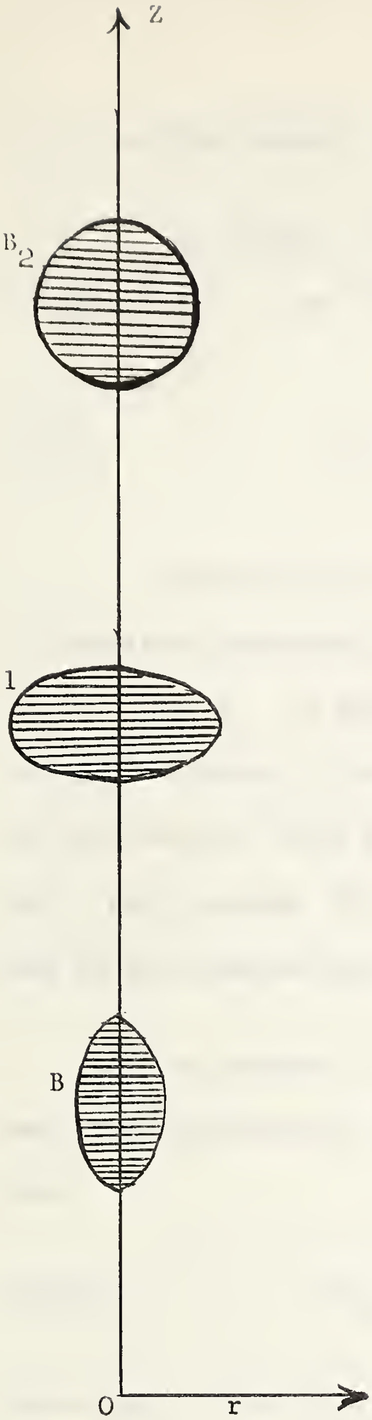

Figure 1.

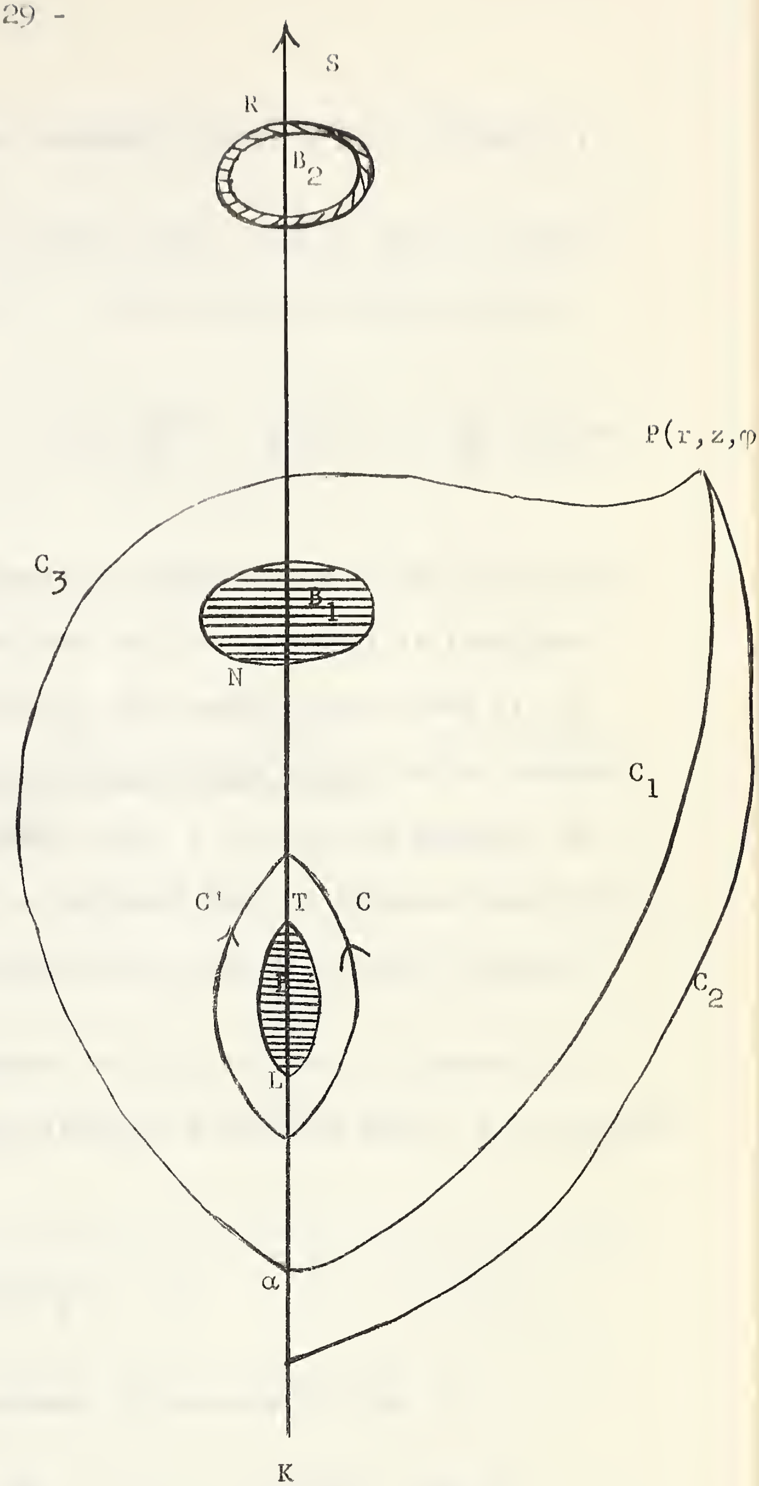

Figure 2.

$\nu = 0$ on the segment KL of z-axis, extending down to $z = -\infty$ (Fig. 2).

To compute $\nu(r, z)$ at any other point $P(r, z, \varphi)$, we connect P to a point α on KL by a path C_1 and evaluate the line integral

$$\nu(r, z)_P = \int_{C_1} r \left\{ \left[\left(\frac{\partial\lambda}{\partial r}\right)^2 - \left(\frac{\partial\lambda}{\partial z}\right)^2 \right] dr + 2\frac{\partial\lambda}{\partial r} \frac{\partial\lambda}{\partial z} dz \right\} . \tag{2.62}$$

Since (2.53) is the integrability condition of (2.54) and (2.55), it therefore expresses the condition that the line-integral is invariant if the path C_1 is deformed continuously into another path (such as C_2 or C_3) joining P to KL <u>without passing through matter</u> in the process of deformation. This proves immediately that $\nu = 0$ on the segment RS . For a path joining KL to RS can be deformed into an infinite semi-circle and (2.61) ensures that the line-integral over this semi-circle vanishes.

To compute ν on the segment ST of the axis, we connect KL and ST by a path C <u>in empty space</u> lying in a meridian plane φ = constant. Then

$$\nu_{ST} = \int_C r(d\underset{\sim}{s} \times \underset{\sim}{A})_\varphi \tag{2.63}$$

where $d\underset{\sim}{s} = (dr, dz, 0)$ is the element of arc-length along C

$$A_r = 2\frac{\partial\lambda}{\partial r} \cdot \frac{\partial\lambda}{\partial z} , \quad A_z = -\left[\left(\frac{\partial\lambda}{\partial r}\right)^2 - \left(\frac{\partial\lambda}{\partial z}\right)^2\right] \tag{2.64}$$

$$A_\varphi = 0$$

and the subscripts r, z, φ denote physical components in the Euclidean map.

Let S be the surface of revolution formed by revolving C about the axis. If $\underset{\sim}{n}$ is the unit outward normal, we have

(2.65) $$r\,d\varphi(\underset{\sim}{ds} \times \underset{\sim}{A})_\varphi = \underset{\sim}{A} \cdot \underset{\sim}{n}\, dS$$

and hence

(2.66) $$\nu_{ST} = \frac{1}{2\pi} \int_C \int_0^{2\pi} r(\underset{\sim}{ds} \times \underset{\sim}{A})_\varphi \, d\varphi$$

(2.67) $$= \frac{1}{2\pi} \iint_S \underset{\sim}{A} \cdot \underset{\sim}{n} \; dS \quad .$$

This integral is independent of the shape of the surface S enclosing the body B : this follows because

(2.68) $$\begin{aligned} \operatorname{div} \underset{\sim}{A} &\equiv \frac{\partial A_r}{\partial r} + \frac{1}{r} A_r + \frac{\partial A_z}{\partial z} \\ &\equiv 2 \frac{\partial \lambda}{\partial z} \nabla^2 \lambda \quad , \end{aligned}$$

which vanishes in matter-free space by virtue of (2.53).

It is easy to show that a function λ , satisfying (2.53), (2.60) and (2.61) in the domain exterior to $B, B_1, B_2, \ldots$ can be uniquely decomposed into a sum of functions

$$\lambda_o + \lambda_1 + \lambda_2 + \ldots .$$

such that λ_i satisfies $\nabla^2 \lambda_i = 0$ everywhere outside B_i .

We now construct a Newtonian analogue system for the bodies B_i by imagining each domain B_i to be filled with a distribution of mass

whose classical Newtonian potential is precisely $\lambda_i\ (r, z)$ exterior to the region B_i . This mass distribution is not unique - e.g. any spherically symmetric distribution with a given total mass has the same external potential.

Converting the surface integral (2.67) into a volume integral taken over the Newtonian analogue of B , we have

$$\nu_{ST} = \frac{1}{2\pi} \iiint_B \operatorname{div} \underset{\sim}{A}\, dv$$

(2.69) $$= \frac{1}{\pi} \iiint_B \frac{\partial\lambda}{\partial z} \nabla^2 \lambda\, dv$$

$$= -4 \iiint_B F_z\, \rho^* \, dv$$

where $\rho^* = \frac{1}{4\pi} \nabla^2\lambda$ is the density of the Newtonian analogue of B , and $F_z = -\frac{\partial\lambda}{\partial z}$ is the z-component of the Newtonian gravitational force field.

We have thus reached the following remarkable result:

The difference between the values of ν on two free segments of the axis of symmetry separated by a material body B is equal (apart from a constant factor) to the Newtonian force exerted on any Newtonian analogue of body B .

It follows that we can satisfy the condition of elementary flatness on every free portion of the axis if and only if we can find a Newtonian analogue system in which each body is in dynamical equilibrium.

From this result, and familiar results of Newtonian theory we can infer immediately that there is no static, axially symmetric solution of Einstein's field equations which corresponds to two bodies on opposite sides of a plane z = constant, each having non-zero total mass. [The relativistic mass m_i of body B_i is defined by

$$(2.70) \qquad - g_{44} = e^{2\lambda_i} = 1 - \frac{2m_i}{\rho} + 0(\frac{1}{\rho^2}) \quad (\rho \to \infty) \ .]$$

On the other hand, static 2-body solutions are possible, if, following Marder, we allow one body to be enclosed by another.

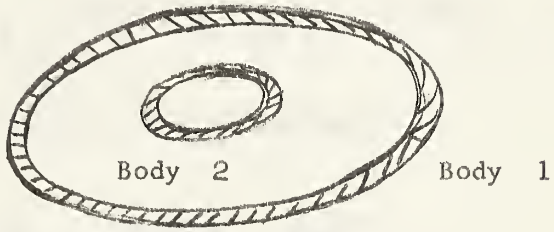

Figure 3.

CHAPTER III

STATIC AXIALLY SYMMETRIC VACUUM FIELDS: PARTICULAR SOLUTIONS

§ 3.1 Introduction.

In the previous chapter, we found that any static, axially symmetric gravitational field in vacuo ("Weyl field") is characterized by the canonical line-element.

$$\Phi = e^{2(\nu-\lambda)}(dr^2+dz^2) + r^2 e^{-2\lambda} d\varphi^2 - e^{2\lambda} dt^2 \quad , \tag{3.1}$$

where the functions $\lambda(r,z)$, $\nu(r,z)$ satisfy

$$\frac{\partial\nu}{\partial r} = r\left[\left(\frac{\partial\lambda}{\partial r}\right)^2 - \left(\frac{\partial\lambda}{\partial z}\right)^2\right] \quad , \tag{3.2a}$$

$$\frac{\partial\nu}{\partial z} = 2r\,\frac{\partial\lambda}{\partial r}\cdot\frac{\partial\lambda}{\partial z} \quad , \tag{3.2b}$$

$$\nabla^2\lambda = \frac{\partial^2\lambda}{\partial r^2} + \frac{1}{r}\frac{\partial\lambda}{\partial r} + \frac{\partial^2\lambda}{\partial z^2} = 0 \quad . \tag{3.3}$$

Equations (3.3) is the integrability condition for (3.2a) and (3.2b).

On segments of the z-axis free of matter, it is necessary that ν satisfy the elementary flatness condition

(3.4) $$\nu(0,z) = 0 \ .$$

For the exterior fields of bounded sources, it is customary to impose the boundary condition of asymptotic flatness at spatial infinity:

(3.5) $$\lambda = 0(\frac{1}{\rho}) \ , \quad \nu = 0(\frac{1}{\rho^2}) \ , \quad (\rho^2 \equiv r^2 + z^2 \to \infty) \ .$$

The paths of infinitesimal test-particles in the field are governed by the geodesic equations

(3.6) $$\frac{d^2x^i}{ds^2} = -\Gamma^i_{pq} \frac{dx^p}{ds} \cdot \frac{dx^q}{ds}$$

where the 4-velocity $\frac{dx^p}{ds}$ is normalized by the identity

(3.7) $$g_{pq} \frac{dx^p}{ds} \cdot \frac{dx^q}{ds} = -1 \ .$$

For particles moving slowly (as compared with the velocity of light) with respect to an observer at rest in the static field, we can neglect squares and products of the spatial components $\frac{dx^1}{ds}$, $\frac{dx^2}{ds}$, $\frac{dx^3}{ds}$. From (3.6) and (3.7) we obtain

(3.8) $$(-g_{44})(\frac{dt}{ds})^2 \approx 1 \ ,$$

$$\frac{d^2x^i}{ds^2} \approx -\Gamma^i_{44} (\frac{dt}{ds})^2$$

(3.9) $$= -\tfrac{1}{2} g^{ij} \frac{\partial \log(-g_{44})}{\partial x^j} \ .$$

At large distances from the sources, we can make the further approximations $g^{ij} \approx \delta^{ij}$, $ds \approx dt$, by virtue of (3.1) and (3.5). Thus, the components of acceleration of a slowly moving test-particle at large distances from the sources of the field (3.1) are

$$(3.10) \qquad \frac{d^2r}{dt^2} \approx -\frac{\partial\lambda}{\partial r}\ , \quad \frac{d^2z}{dt^2} \approx -\frac{\partial\lambda}{\partial z}\ , \quad \frac{d^2\varphi}{dt^2} \approx 0\ .$$

The conditions under which (3.10) is a valid approximation are precisely those for which Newton's theory should give an accurate description. We are thus led to conclude that: at large distances from the sources, where the Newtonian approximation is meaningful, the function $\lambda(r,z)$ is identical with the Newtonian potential of the distribution. We note further the remarkable fact that in Weyl's canonical co-ordinates, $\lambda(r,z)$ satisfies throughout free space the equation (3.3), which is formally Laplace's equation in cylindrical co-ordinates.

These facts suggest the following very convenient method of generating and "labelling" particular Weyl fields.

(i) Within the framework of Newton's theory, consider an axially symmetric distribution of matter $\mathcal{D}$ having the exterior Newtonian potential $V(r,z)$.

(ii) Set $\lambda(r,z) = V(r,z)$. Then (3.3) is satisfied. Compute $\nu(r,z)$ from (3.2).

We thus obtain a Weyl field (3.1) which at large distances is indistinguishable from the predictions of Newton's theory concerning the

external field of the distribution $\mathcal{D}$ in Euclidean space.

At smaller distances, we have to reckon with distortion of the geometry caused by the gravitational field. Thus, we might set λ formally equal to the Newtonian potential of a spheroid and label the resulting Weyl field as that of a "spheroid", but as a description this is only good at large distances. In a curved space, we experience difficulty in even defining a spheroid. The point emerges most strikingly in the case of spherical symmetry. If we start from $\lambda = - m/(r^2+z^2)^{\frac{1}{2}}$, corresponding to a spherically symmetric Newtonian distribution $\mathcal{D}$, the resulting line-element (3.1) is not spherically symmetric (see § 3.3), although it approximates to spherical symmetry at large distances. To obtain the Schwarzschild spherically symmetric solution for a particle, one has to set λ formally equal to the potential of a rod, as Weyl already pointed out in his fundamental paper of 1917 [3]. Each Weyl field generated by steps (i) and (ii) ought to be re-examined a posteriori to discover what distortion of the originally contemplated Newtonian distribution is produced by the curvature of space. Such a re-examination involves difficulties both of principle and of practice, and has been carried out only in very few cases, such as spherical symmetry. For the most part we shall be content in this chapter with the uncritical procedure of generating and "labelling" particular Weyl fields according to steps (i) and (ii). An initial assault on the problem of intrinsic characterization of Weyl fields is due to Ehlers and Kundt [44,45].

§ 3.2 Fields of unbounded distributions of mass.

We begin our catalogue of special solutions with some physically unrealistic situations in which the distribution of matter extends to infinity and the condition of asymptotic flatness (3.5) is violated.

In an attempt to reproduce the field of an infinite plane in general relativity, it seems natural to start from the Newtonian potential $V = gz$, where g is a constant. Applying steps (i) and (ii), we obtain after an elementary calculation

$$\lambda = gz \quad , \quad \nu = -\tfrac{1}{2}g^2r^2 \quad . \tag{3.11}$$

However, the Weyl field (3.1) with λ, ν given by (3.11) does not have the property of homogeneity which ought to characterize the field of an infinite plane; i.e., it does not admit a two-parameter group of spatial translations orthogonal to the axis of symmetry. A Weyl field having this property is given by

$$\lambda = \tfrac{1}{2}\log \frac{\rho+z}{2\beta} \quad , \quad \nu = \tfrac{1}{2}\log \frac{\rho+z}{2\rho} \tag{3.12a}$$

where

$$\rho^2 \equiv r^2 + z^2$$

and β is a constant. It is easily verified that this is a solution of (3.2), (3.3). The conformal transformation of the space co-ordinates

$$2\beta r = r_* z_* \ , \quad 2\beta z = \tfrac{1}{2}(z_*^2 - r_*^2) \tag{3.13}$$

converts the line-element (3.1), (3.12a) into

$$\Phi = dr_*^2 + r_*^2\, d\varphi^2 + dz_*^2 - \left(\frac{z_*}{\beta}\right)^2 dt^2 \quad . \tag{3.12b}$$

In this form, the homogeneity of the manifold is directly evident.

The spatial transformation

$$\tfrac{1}{4}g^2\, z_*^2 = 1 + g\bar{z} \tag{3.14}$$

combined with the scale-transformation of time $\bar{t} = \frac{2}{g}\, t$ throws (3.12b) into the form

$$\Phi = dr_*^2 + r_*^2\, d\varphi^2 + \frac{d\bar{z}^2}{1+g\bar{z}} - (1+g\bar{z})\, d\bar{t}^2 \quad . \tag{3.12c}$$

The gravitational field of an "infinite plane" was obtained in the form (3.12c) by Chou [25] in 1931, using a somewhat different approach, and still earlier by Whittaker [46] who derived it as a limit of the Schwarzschild solution when the gravitating centre is removed to infinity and its mass increased proportionately [cf. Chapter V].

Neither Whittaker nor Chou seemed aware that (3.12) represents a flat manifold. To see this, consider the Minkowskian line-element

$$\Phi = d\xi^2 + d\eta^2 + d\zeta^2 - d\tau^2$$

and apply the transformation

$$\begin{aligned} \xi &= r_* \cos\varphi \;, & \eta &= r_* \sin\varphi \\ \zeta &= z_* \sinh\left(\frac{t}{\beta}\right) \;, & \tau &= z_* \cosh\left(\frac{t}{\beta}\right) \;; \end{aligned} \tag{3.15}$$

we then obtain (3.12b). We observe from (3.15) that each point with fixed space co-ordinates r_* , z_* , φ executes hyperbolic motion: ξ = constant , η = constant , $\zeta^2 - \tau^2$ = const. Hence (3.12b) may be interpreted as the metric of flat space-time viewed from a uniformly accelerated reference frame or (according to the principle of equivalence) as a homogeneous gravitational field. The line-elements (3.12a) and (3.12c) are open to the same interpretation, since they are connected with (3.12b) by purely spatial transformations which do not affect the motion of the frame.

We turn to what could be labelled the field of an infinite rod or cylinder:

$$\lambda = 2m \log \frac{r}{a} , \quad \nu = 4m^2 \log \frac{r}{a} . \tag{3.16}$$

A result equivalent to this was given by Wilson [19] in 1920, and before him by Levi-Civita [4] in 1918.

In the case of a hollow cylindrical shell of radius a , the exterior metric (3.16) can be joined to the flat interior solution

$$\Phi = dr^2 + dz^2 + r^2 d\varphi^2 - dt^2$$

with continuity of the metric tensor g_{ij} . The jump discontinuity in the first derivatives,

$$\left[\frac{\partial\lambda}{\partial r}\right]_{r=a_+} - \left[\frac{\partial\lambda}{\partial r}\right]_{r=a_-} = \frac{2m}{a} , \tag{3.17}$$

gives, analogously to Newtonian formula

$$(3.18) \qquad \left(\frac{\partial V}{\partial n}\right)_+ - \left(\frac{\partial V}{\partial n}\right)_- = 4\pi\sigma$$

a measure of the surface density of the shell. Comparison of (3.17) and (3.18) appears to indicate that m is to be interpreted as the mass of the shell (in gravitational units) per unit length. This can also be checked by looking at the geodesics (paths of test-particles) in the manifold (3.1), (3.16). For slowly moving particles [cf. (3.9)], we have

$$(3.19) \qquad \frac{d^2\varphi}{ds^2} \approx 0, \quad \frac{d^2 z}{ds^2} \approx 0 , \quad \frac{d^2\bar{r}}{ds^2} \approx -\frac{2m}{r} ,$$

where $d\bar{r} \equiv \sqrt{g_{11}}\, dr$ is the element of arc-length along the parametric lines of r. Equation (3.19) is in agreement with the Newtonian result for the acceleration due to an infinite cylinder of mass m per unit length.

§ 3.3 Static one-body solution: Curzon and Schwarzschild particles.

The formally simplest solution of the potential equation $\nabla^2\lambda = 0$ which satisfies the asymptotic boundary condition (3.5) corresponds to the field of a monopole in the Newtonian analogue:

$$(3.20) \qquad \lambda = -\frac{m}{\rho} , \quad \text{leading to} \quad \nu = -\frac{m^2 r^2}{2\rho^4} ,$$

where, as usual, $\rho^2 = r^2 + z^2$. The manifold (3.1), (3.20) is <u>not</u> spherical symmetric [47], i.e., it is not related to the Schwarzschild solution by a co-ordinate transformation. We shall refer to it as the field of a <u>Curzon</u> particle. The curious nature of the singularity at $\rho = 0$ has been remarked upon by Mysak and Szekeres [48], who have studied it intrinsically by the introduction of geodesic co-ordinates.

The Weyl field

$$\lambda = \frac{m}{b} \log \frac{\rho + \rho' - b}{\rho + \rho' + b}$$

(3.21)

$$\nu = 2\frac{m^2}{b^2} \log \frac{(\rho+\rho')^2 - b^2}{4\rho\rho'} ,$$

where

$$\rho^2 = r^2 + z^2 \quad , \quad \rho'^2 = r^2 + (z - b)^2 \quad , \tag{3.22}$$

has as its Newtonian analogue the field of a <u>uniform rod</u> of mass m extending along the axis of symmetry from $z = 0$ to $z = b$ [49] . One readily verifies that the elementary flatness condition, $\nu = 0$ when $r = 0$, is satisfied for z outside the range $(0,b)$.

If we set $m = \frac{1}{2}b$, (3.21) is equivalent to the well-known <u>Schwarzschild spherically symmetric solution</u>. The co-ordinate transformation $(r,z) \to (R,\theta)$, defined by

$$r = R(1 - \frac{b}{R})^{\frac{1}{2}} \sin\theta \ , \ z - \tfrac{1}{2}b = (R - \tfrac{1}{2}b) \cos\theta \tag{3.23}$$

reduces (3.1), (3.2), (3.21), (with $m = \frac{1}{2}b$) to

$$(3.24a) \qquad \Phi = \frac{dR^2}{1-\frac{b}{R}} + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - (1 - \frac{b}{R})dt^2$$

(see Appendix II)

A discussion of this line-element is included in every standard text, and we need not dally over it here. However, there is one remark which should be made, because it will be important in our later discussions. It has been shown by many authors [50, 51, 52, 53], that the singularity of the form (3.24) at $R = b$ is not a geometrical singularity of the manifold, but it is due to the nature of the co-ordinates used. Kruskal [54] has exhibited a co-ordinate system which covers the entire Schwarzschild manifold without singularity, apart from the geometric singularity at $R = 0$. Consider the line-element

$$(3.24b) \qquad \Phi = \frac{d\zeta^2 - d\tau^2}{(1+p)e^p} + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

where R and p are functions of $\zeta^2 - \tau^2$ defined by

$$(3.25) \qquad R = b(1+p) \quad , \quad 4b^2pe^p = \zeta^2 - \tau^2 \quad .$$

This line-element is regular in the domain $\zeta^2 - \tau^2 > -\frac{4b^2}{e}$. If we make the co-ordinate transformation $(\zeta,\tau) \to (R,t)$ where t is defined by

$$(3.26) \qquad \tanh\left(\frac{t}{2b}\right) = \frac{\tau}{\zeta}$$

on the subdomain $\zeta^2 - \tau^2 > 0$, we obtain the Schwarzschild form (3.24a). Kruskal's manifold (3.24b) is therefore an analytic extension of the Schwarzschild manifold. It can be shown that the Kruskal manifold is <u>maximal</u>, in the sense that every geodesic either runs into the proper

singularity $R = 0$ or else can be continued to infinite values of its geodesic arc parameter. It is further immediately clear that (3.24b) represents a solution of Einstein's vacuum equations $R_{ij} = 0$ throughout its domain of validity $\zeta^2 - \tau^2 > - \frac{4b^2}{e}$ since

(i) we certainly have $R_{ij} = 0$ for the subdomain of the Schwarzschild metric $\zeta^2 - \tau^2 > 0$;

(ii) R_{ij} is an analytic function of the Kruskal co-ordinates in the region $\zeta^2 - \tau^2 > - \frac{4b^2}{e}$.

A detailed and very interesting discussion of Kruskal's manifold has recently been given by Fuller and Wheeler [55].

Oblate spheroidal shells.

Chou [25] and Misra [27] have obtained the Weyl field whose Newtonian analogue is a spheroidal homeoid, i.e., a thin spheroidal shell whose surface distribution of mass is such that the potential over its surface is constant.

Let us consider the Newtonian situation for a moment. We recall that oblate spheroidal co-ordinates ξ , η , φ , are defined by

$$(3.27) \qquad r = b\sqrt{(1+\xi^2)(1-\eta^2)} \quad , \quad z = b\xi\eta \quad , \varphi = \varphi$$

in terms of cylindrical co-ordinates, r, z, φ . Spheroidal co-ordinates form an orthogonal net, as one immediately sees on writing (3.27) in

complex form:

$$r + iz = b \cosh[\sinh^{-1}\xi + i \sin^{-1} \eta] \quad . \tag{3.28}$$

This shows that the transformation $(r,z) \rightarrow (\sinh^{-1}\xi, \sin^{-1} \eta)$ is conformal. The surfaces η = constant are the hyperboloids of one sheet

$$\frac{r^2}{b^2(1-\eta^2)} - \frac{z^2}{b^2\eta^2} = 1 \quad . \tag{3.29}$$

The surfaces ξ = constant are the oblate spheroids

$$\frac{r^2}{b^2(1+\xi^2)} + \frac{z^2}{b^2\xi^2} = 1 \quad . \tag{3.30}$$

In particular, the surface $\xi = 0$ is the interior of a disk of radius b in the xy plane: $r = 0$, $z \leq b$, and the surface $\eta = 0$ is the region of the xy plane exterior of this same disk: $r = 0$, $z \geq b$. The surfaces $\eta = +1$ and $\eta = -1$ are respectively the positive and negative segments of the z-axis.

In terms of oblate spheroidal co-ordinates, the Laplacian reduces to [56]

$$\nabla^2 V = \frac{1}{b^2(\xi^2+\eta^2)} \left[\frac{\partial}{\partial\xi} (1+\xi^2) \frac{\partial V}{\partial\xi} + \frac{\partial}{\partial\eta}(1-\eta^2) \frac{\partial V}{\partial\eta} \right] \tag{3.31}$$

for an axially symmetric function $V(\xi,\eta)$. Separable solutions of Laplace's equation $\nabla^2 V = 0$ may be shown to be of the form

$$V(\xi,\eta) = [a_n P_n(i\xi) + b_n Q_n(i\xi)][\alpha_n P_n(\eta) + \beta_n Q_n(\eta)] \quad . \tag{3.32}$$

Here, P_n is the n^{th} Legendre polynomial and Q_n the Legendre function of the second kind, defined by

$$Q_n(\mu) = P_n(\mu) \int_\mu^\infty \frac{d\mu}{(\mu^2-1)P_n^2} \quad . \tag{3.33}$$

Particular values are

$$\begin{cases} P_o(\mu) = 1, \quad P_1(\mu) = \mu \;, \; P_2(\mu) = \frac{1}{2}(3\mu^2-1), \; \ldots. \\ Q_o(\mu) = \frac{1}{2}\log\frac{\mu+1}{\mu-1} = \coth^{-1}\mu = \frac{1}{i}\cot^{-1}(\frac{\mu}{i}) \\ Q_1(\mu) = \frac{1}{2}\mu\log\frac{\mu+1}{\mu-1} - 1 \\ Q_2(\mu) = \frac{1}{4}(3\mu^2-1)\log\frac{\mu+1}{\mu-1} - \frac{3}{2}\mu \end{cases} \tag{3.34}$$

To obtain a vacuum potential which is constant over one of the spheroids (3.30), we have to choose a function (3.32) which depends on ξ only, and tends to zero at spatial infinity $(\xi \to \infty)$. We thus arrive at the <u>Newtonian potential of an oblate spheroidal homeoid</u>:

$$V = \frac{m}{ib} Q_o(i\xi)\, P_o(\eta) = -\frac{m}{b}\cot^{-1}(\xi) \; . \tag{3.35}$$

At large distances we have from (3.27)

$$\rho^2 = r^2 + z^2 \approx b^2\xi^2 \;, \quad \cot^{-1}\xi \approx \frac{1}{\xi} \;, \quad (\xi \to \infty) \tag{3.36}$$

and hence

$$V \approx -\frac{m}{\rho} \qquad (\rho \to \infty)$$

showing that the constant m in (3.35) is the total mass of the spheroidal shell.

To obtain the corresponding Weyl field, it is convenient to re-write (3.2) in terms of spheroidal co-ordinates. This is done in Appendix III , and similar calculations have been carried out by Misra [27].

The result is

$$(3.37a) \quad \frac{\partial \nu}{\partial \xi} = \frac{(1+\xi^2)(1-\eta^2)}{\xi^2+\eta^2}\left(\frac{\partial \lambda}{\partial \xi}\right)^2 - \frac{\xi(1-\eta^2)^2}{\xi^2+\eta^2}\left(\frac{\partial \lambda}{\partial \eta}\right)^2 - \frac{2\eta(1+\xi^2)(1-\eta^2)}{\xi^2+\eta^2}\frac{\partial \lambda}{\partial \xi}\cdot\frac{\partial \lambda}{\partial \eta}$$

$$(3.37b) \quad \frac{\partial \nu}{\partial \eta} = \frac{\eta(1+\xi^2)^2}{\xi^2+\eta^2}\left(\frac{\partial \lambda}{\partial \xi}\right)^2 - \frac{\eta(1-\eta^2)(1+\xi^2)}{(\xi^2+\eta^2)}\left(\frac{\partial \lambda}{\partial \eta}\right)^2 + \frac{2(1-\eta^2)\xi(1+\xi^2)}{(\xi^2+\eta^2)}\frac{\partial \lambda}{\partial \xi}\cdot\frac{\partial \lambda}{\partial \eta}$$

With the aid of these formulae, we easily obtain the <u>Weyl field whose Newtonian analogue is the field of an oblate spheroidal homeoid</u>:

$$(3.38) \qquad \lambda = -\frac{m}{b}\cot^{-1}\xi \quad , \quad \nu = \frac{m^2}{2b^2}\log\frac{\xi^2+\eta^2}{1+\xi^2} \quad .$$

Let us now consider the question of joining this solution to a Weyl manifold representing the gravitational field in the hollow interior of the spheroidal shell $\xi = \xi_o$ [See (3.30).]. If we impose the reasonable requirement that λ is to be continuous across the shell, then the interior solution is

(3.39) $$\lambda = -\frac{m}{b}\cot^{-1}\xi_o \ , \quad \nu = 0 \ ,$$

since the only harmonic function λ which is constant over a closed surface is constant everywhere inside. We observe from (3.38) and (3.39) that:

(i) The elementary flatness condition, $\nu = 0$ when $r = 0$, is satisfied everywhere on the axis $\eta^2 = 1$.

(ii) ν is discontinuous across the shell - i.e. Weyl's canonical co-ordinates are "inadmissible" in the sense that the metric tensor experiences jump discontinuities across a surface distribution of mass.

(iii) If we replace ξ by $\frac{\xi}{\epsilon}$, b by $b\epsilon$ and let $\epsilon \to 0$, we obtain from (3.30) and (3.27)

(3.40) $$\rho = b\xi \ , \quad r = \rho\sqrt{1-\eta^2} \ , \quad z = \rho\eta$$

so that we obtain in the limit spherical polar co-ordinates ρ, $\cos^{-1}\eta$, φ in the Euclidean map. In this limit, (3.38) and (3.39) become

(3.41) $$\begin{cases} \lambda = -\frac{m}{\rho} \ , \quad \nu = -\frac{m^2}{2\rho^2}(1-\eta^2) & \text{(Exterior)} \\ \lambda = -\frac{m}{\rho_o} \ , \quad \nu = 0 \ . & \text{(Interior)} \end{cases}$$

These results should be compared with our solution (3.20) for the field of a "Curzon particle". The field (3.41) might be called a "Curzon Shell".

(iv) Space-time is flat inside a homeoid. This is in harmony with Newtonian theory, according to which the interior gravitational field vanishes.

Prolate spheroidal shells.

The replacements $b \to ib$, $\xi \to \frac{1}{i}\xi$ convert (3.27) and (3.29) to

(3.42) $$r = b\sqrt{(1-\xi^2)(\eta^2-1)} \quad , \quad z = b\xi\eta \quad ,$$

(3.43) $$\frac{r^2}{b^2(\eta^2-1)} + \frac{z^2}{b^2\eta^2} = 1 \quad .$$

The surfaces η = constant are prolate spheroids. Solutions of Laplace's equation have the form

(3.44) $$V = [a_n P_n(\xi)+b_n Q_n(\xi)][\alpha_n P_n(\eta)+\beta_n Q_n(\eta)] \quad .$$

The Weyl field

(3.45) $$\begin{cases} \lambda = \frac{m}{b} Q_o(\eta) = \frac{m}{2b} \log \frac{\eta-1}{\eta+1} \\ \nu = \frac{m^2}{2b^2} \log \frac{\eta^2-1}{\eta^2-\xi^2} \end{cases}$$

corresponds to a Newtonian field whose potential is constant on the surfaces η = constant. It may therefore be labelled the field of a prolate spheroidal homeoid. Noting that

(3.46) $$\eta = \frac{\rho_1 + \rho_2}{2b} \quad , \quad \xi = \frac{\rho_2 - \rho_1}{2b} \quad ,$$

where

(3.47) $$\rho_1^2 = r^2 + (z-b)^2 \ , \quad \rho_2^2 = r^2 + (z+b)^2 \ ,$$

and comparing with (3.21), we recognize (3.45) as representing also the field of a <u>uniform rod</u> of mass m and extending from $z = -b$ to $z = b$ along the axis of symmetry.

Two special cases are of interest.

(i) Choosing $m = \frac{1}{2}b$ we obtain the Schwarzschild solution, representing the exterior field of a spherical shell.

(ii) Replace b by ϵb and η by $\frac{\eta}{\epsilon}$, and let $\epsilon \to 0$. We then obtain, as before, the external field of a Curzon shell.

<u>Solid spheroids</u>.

The Weyl manifolds whose Newtonian analogues are the exterior fields of solid homogeneous spheroids have recently been given by Misra [27], and were independently computed by the author. Details of the calculations are given in Appendix IV. Here we quote only the results.

For an <u>oblate homogeneous spheroid</u>

(3.48a) $$\lambda = A[4iQ_o(i\xi)P_o(\eta) - 4iQ_2(i\xi)P_2(\eta)]$$

(A is constant)

$$= A[4\cot^{-1}\xi + (3\eta^2-1)\{(1+3\xi^2)\cot^{-1}\xi - 3\xi\}]$$

leading to

$$(3.48b) \quad \nu = A^2\eta^4[(1+\xi^2)(1+9\xi^2)(\cot^{-1}\xi)^2-(14\xi+18\xi^3)\cot^{-1}\xi+9\xi^2+4]$$

$$- 2A^2\eta^2[(1+\xi^2)(1+5\xi^2)(\cot^{-1}\xi)^2-(6\xi+10\xi^3)\cot^{-1}\xi+5\xi^2]$$

$$+ A^2[(1+\xi^2)(\cot^{-1}\xi)^2+2(\xi-\xi^3)\cot^{-1}\xi+\xi^2-4] \quad .$$

<u>For a prolate homogeneous spheroid</u>

$$(3.49a) \quad \lambda = A[4P_0(\xi)Q_0(\eta) - 4P_2(\xi)Q_2(\eta)]$$

$$= A[4 \coth^{-1}\eta - (3\xi^2-1)\{(3\eta^2-1)\coth^{-1}\eta - 3\eta]$$

leading to

$$(3.49b) \quad \nu = A^2\xi^4[(\eta^2-1)(9\eta^2-1)(\coth^{-1}\eta)^2 + (14\eta-18\eta^3)\coth^{-1}\eta+9\eta^2-4]$$

$$+ 2A^2\xi^2[(\eta^2-1)(1-5\eta^2)(\coth^{-1}\eta)^2 +(10\eta^3-6\eta)\coth^{-1}\eta-5\eta^2]$$

$$+ A^2[(\eta^2-1)^2(\coth^{-1}\eta)^2 - 2(\eta^3+\eta)\coth^{-1}\eta + \eta^2 + 4] \quad .$$

<u>Field of a ring</u>:

One of the first Weyl fields given explicitly was the field of a ring, worked out by Bach [8] in 1922. For details, we refer to Bach's paper.

<u>Field of a mass dipole</u>:

The following result is new.

For λ take the harmonic function

(3.50a) $$\lambda = \frac{mz}{\rho^3}$$

leading to

(3.50b) $$\nu = -\frac{2m^2r^2}{\rho^6} + \frac{9}{4}\frac{m^2r^4}{\rho^8}$$ (See Appendix V) .

Because of the peculiar character of the "force" between a free positive and a free negative mass in general relativity (see Chapter V for a discussion) the internal mechanical stresses holding this dipole in equilibrium must be of a quite remarkable nature: a pressure at the negative pole and a tension at the positive pole.

Erez and Rosen [26] have given the field of a mass quadrupole

(3.51a) $$\lambda = \frac{1}{2}\{[1 + \frac{1}{4}q(3\eta^2-1)(3\xi^2-1)] \log\frac{\eta-1}{\eta+1} + \frac{3}{2}q\eta(3\xi^2-1)\}$$

(3.51b) $$\begin{aligned}\nu = {} & \frac{9}{64}q^2\xi^4\{(9\eta^4-10\eta^2+1)(\log\frac{\eta-1}{\eta+1})^2 + (36\eta^3-28\eta)\log\frac{\eta-1}{\eta+1} + 36\eta^2-16\} \\ & + \xi^2\{\frac{3}{32}q^2(-5\eta^4+6\eta^2-1)(\log\frac{\eta-1}{\eta+1})^2 + [\frac{3}{2}q\eta + \frac{9}{32}q^2(-20\eta^3+\frac{52}{3}\eta)]\log\frac{\eta-1}{\eta+1} \\ & + 3q + \frac{9}{32}q^2(-20\eta^2+\frac{32}{3})\} + (\frac{1}{2}q^2+q+\frac{1}{2})\log\frac{\eta^2-1}{\eta^2-\xi^2} \\ & + \frac{9}{16}q^2(\eta^4-2\eta^2+1)(\log\frac{\eta-1}{\eta+1})^2 + \{\frac{1}{16}q^2(9\eta^2-15\eta)-\frac{3}{2}q\eta\}\log\frac{\eta-1}{\eta+1} \\ & + \frac{9}{16}q^2(\eta^2+1)+3q\end{aligned}$$

where q is an arbitrary constant.

§ 3.4 Field of n bodies: general formulae.

In the case of a Weyl manifold whose Newtonian analogue is the static field of n distinct bodies $B_1, B_2, \ldots, B_n$, we may set

$$(3.52)\qquad \lambda = \sum_{i=1}^{n} \lambda_i$$

where λ_i is the external Newtonian potential of B_i, and thus satisfies

$$(3.53)\qquad \nabla^2 \lambda_i = 0 .$$

To obtain ν from (3.2), we compute ν_{ij} from

$$(3.54)\qquad \begin{aligned} \frac{\partial \nu_{ij}}{\partial r} &= 2r\left(\frac{\partial \lambda_i}{\partial r}\frac{\partial \lambda_j}{\partial r} - \frac{\partial \lambda_i}{\partial z}\frac{\partial \lambda_j}{\partial z}\right) \\ \frac{\partial \nu_{ij}}{\partial z} &= 2r\left(\frac{\partial \lambda_i}{\partial r}\frac{\partial \lambda_j}{\partial z} + \frac{\partial \lambda_i}{\partial z}\frac{\partial \lambda_j}{\partial r}\right) \end{aligned}$$

and then

$$(3.55)\qquad \nu = \sum_{i=1}^{n} \sum_{j=1}^{n} \nu_{ij} .$$

The integrability condition of (3.54) is

$$\frac{\partial \lambda_i}{\partial z} \nabla^2 \lambda_j + \frac{\partial \lambda_j}{\partial z} \nabla^2 \lambda_i = 0$$

and is satisfied by virtue of (3.53).

§ 3.5 Static n-body solutions: formal results.

We begin by deriving a formal n-body solution of the equations (3.53) and (3.54), deferring until later the question of physical interpretation.

Let

$$(3.56) \qquad \lambda_i = \frac{m_i}{b_i} \log \frac{\rho_i + \rho_i' - b_i}{\rho_i + \rho_i' + b_i}$$

where

$$(3.57) \qquad \rho_i^2 = r^2 + z_i^2 \ , \quad \rho_i'^2 = r^2 + z_i'^2 \ ,$$

$$(3.58) \qquad z_i = z - (a_i + \tfrac{1}{2}b_i) \ , \quad z_i' = z - (a_i - \tfrac{1}{2}b_i) \ .$$

Formally, $\lambda = \Sigma\lambda_i$ represents the Newtonian potential of a set of collinear rods on the axis of symmetry, the i^{th} rod extending from $z = a_i - \frac{1}{2}b_i$ to $z = a_i + \frac{1}{2}b_i$, and having mass m_i. It will be convenient to define

$$E(i,j) = \rho_i\rho_j + z_i z_j + r^2$$

$$(3.59) \qquad E(i',j) = E(j,i') = \rho_i'\rho_j + z_i' z_j + r^2$$

$$E(i',j') = \rho_i'\rho_j' + z_i' z_j' + r^2 \ .$$

The integration of (3.54) is easily performed with the aid of identities of the type

$$d \log[r^{-1}E(i,j')] = \frac{r^2 - z_i z_j'}{r\rho_i\rho_j'}\,dr + \frac{z_i + z_j'}{\rho_i\rho_j'}\,dz \tag{3.60}$$

(for details see Appendix VI), we obtain

$$\nu_{ij} = \frac{m_i m_j}{b_i b_j} \log \frac{E(i',j)E(i,j')}{E(i,j)E(i',j')} \; . \tag{3.61}$$

We have thus obtained a Weyl manifold representing n bodies, given by (3.52), (3.55), (3.56) and (3.61). This result is given here for the first time.

If in (3.56) and (3.61), we let each $b_i \to 0$, we obtain in the limit a solution which is formally the field of n collinear Curzon particles:

$$\left\{\begin{aligned} \lambda &= -\sum_{i=1}^{n} \frac{m_i}{\rho_i} \\ \nu &= \sum_{i=1}^{n}\sum_{j=1}^{n} \nu_{ij} \\ \nu_{ij} &= \frac{m_i m_j}{\rho_i\rho_j}\,\frac{z_i z_j + r^2 - \rho_i\rho_j}{(z_i - z_j)^2} \qquad (i \neq j) \\ \nu_{ii} &= -\tfrac{1}{2}r^2\,\frac{m_i^2}{\rho_i^4} \end{aligned}\right. \tag{3.62}$$

(details are given in Appendix VII). A solution equivalent to this was given by Hoffmann [21]. For $n=2$ this solution was given by Curzon [12]

in 1924, by Silberstein [13] in 1936, and by Chazy [10] in 1923.

To Chazy is due also a form of the solution representing two Schwarzschild particles, and equivalent forms were given by Palatini [11] and (earliest of all) by Bach [8]. Bach gave his solution in essentially the following form:

$$(3.63)\quad \begin{cases} \lambda = \displaystyle\sum_{i=1}^{2} \tfrac{1}{2}\log \frac{\rho_i + \rho_i' - b_i}{\rho_i + \rho_i' + b_i} \\[2ex] \nu = \displaystyle\sum_{i=1}^{2} \tfrac{1}{2}\log \frac{(\rho_i+\rho_i')^2 - b_i^2}{4\rho_i\rho_i'} + \log\frac{c(c+b_2)\rho_1 + c(c+b_1+b_2)\rho_1' - cb_1\rho_2'}{c(c+b_2)\rho_1 + (c+b_1)(c+b_2)\rho_1' - b_1(c+b_2)\rho_2} \end{cases}$$

where

$$(3.64)\quad \tfrac{1}{2}c = a_2 - a_1 + \tfrac{1}{2}(b_2 - b_1) .$$

Chazy's 2-body solution was expressed in terms of spheroidal co-ordinates. Define x_1, y_1 and x_2, y_2 by

$$(3.65)\quad r + i(z-a_1) = \sinh(x_1+iy_1) , \; r + i(z-a_2) = \sinh(x_2+iy_2) .$$

The variables x_i, y_i are related to our prolate spheroidal co-ordinates of (3.42) by

$$\eta = \cosh x_i \;, \quad \xi = \sin y_i \quad (\text{assuming } a_i = 0)$$

Chazy takes [cf.(3.45)]

(3.66a) $$\lambda = k_1 \log \tanh \frac{x_1}{2} + k_2 \log \tanh \frac{x_2}{2}$$

and finds

(3.66b) $$\nu = \sum_{i=1}^{2} \frac{k_i^2}{2} \log \frac{\sinh^2 x_i}{\sinh^2 x_i + \cos^2 y_i} + k_1 k_2 \log\left[1 + \frac{2}{a_1 - a_2} \frac{\sin y_1 - \sin y_2}{\cosh x_1 + \cosh x_2}\right]$$

All the results of this section, both for Schwarzschild and for Curzon-type bodies, are of course, subsumed in our general formulas (3.56), (3.61). In what follows we shall therefore concentrate our attention on these.

§ 3.6 Field of n bodies: the elementary flatness condition.

With the solitary exception of Bach, the authors of the formal two-body solutions, we have reviewed in the previous section consistently misinterpreted their results as the field of two particles at rest on the axis of symmetry. This was because they ignored the vital condition of elementary flatness on the axis of symmetry. We shall now examine the consequences of this condition for the general formal n-body solution (3.56), (3.61).

We shall proceed on the assumption that the field (3.56), (3.61) is produced by an axially symmetric system of n-bodies, each of which is separated from the others by an empty region completely (no struts).

Let us also assume, for the moment, that the n rods in the Euclidean map be arranged in non-overlapping sequence along the z-axis:

$$b_i > 0\ ,\quad a_i + \tfrac{1}{2}b_i < a_{i+1} - \tfrac{1}{2}b_{i+1} \qquad (i = 1,2,3,\ldots,n-1) \tag{3.67}$$

(This may not be possible if the $(i+1)^{th}$ body is hollow and encloses the i^{th} body).

Elementary flatness requires that $\nu = 0$ on sections of the z-axis which are free of material singularities. According to (3.54), we have $\frac{\partial \nu_{ij}}{\partial z} = 0$ and hence ν_{ij} = constant on segments of the z-axis not occupied by the i^{th} or j^{th} rods in the Euclidean map. Explicitly,

$$\nu_{ij} = 0 \quad (r=0,\ i \leq j,\ z > a_j+\tfrac{1}{2}b_j \ \text{ or } \ z < a_i-\tfrac{1}{2}b_i)$$

$$\nu_{ij} = \nu^{(o)}_{ij} = 2\frac{m_i m_j}{b_i b_j} \log \left| \frac{(z_i-z'_j)(z'_i-z_j)}{(z_i-z_j)(z'_i-z'_j)} \right| \tag{3.68}$$

$$(r = 0,\ i < j\ ,\ a_i+\tfrac{1}{2}b_i < z < a_j-\tfrac{1}{2}b_j)$$

(details are given in appendix VIII).

Thus the elementary flatness imposes $(n-1)$ discrete conditions. These can be written down by expressing the fact that the jump in ν when crossing the i^{th} rod is zero:

$$\sum_{j=1}^{i-1} \nu^{(o)}_{ij} - \sum_{j=i+1}^{n} \nu^{(o)}_{ij} = 0\ . \tag{3.69}$$

In the Euclidean map, $\nu^{(o)}_{ij}$ represents precisely the Newtonian force between the i^{th} and j^{th} rods [Appendix IX, the result is also obvious

from our general theorem of Chapter II, page 32]. Hence (3.69) may be interpreted as the relativistic analogue of the condition that the resultant gravitational force on the i^{th} body is zero.

For <u>two</u> bodies, (3.69) reduces to the single condition

$$\frac{m_1 m_2}{b_1 b_2} \log\left[1 - \frac{b_1 b_2}{(a_1-a_2)^2 - \frac{1}{4}(b_1-b_2)^2}\right] = 0 ,$$

which cannot be satisfied for any non-zero values of the masses m_1, m_2.

For <u>three</u> bodies, (3.69) becomes

$$\nu_{12}^{(o)} = \nu_{23}^{(o)} = -\nu_{13}^{(o)}$$

or

$$\frac{m_1 m_2}{b_1 b_2} \log\left[1 - \frac{b_1 b_2}{(a_1-a_2)^2 - \frac{1}{4}(b_1-b_2)^2}\right] \tag{3.70}$$

$$= \frac{m_2 m_3}{b_2 b_3} \log\left[1 - \frac{b_2 b_3}{(a_2-a_3)^2 - \frac{1}{4}(b_2-b_3)^2}\right]$$

$$+ \frac{m_1 m_3}{b_1 b_3} \log\left[1 + \frac{b_1 b_3}{(a_1-a_3)^2 - \frac{1}{4}(b_1+b_3)^2}\right] .$$

If we insist that all masses are to be positive, then these conditions cannot be satisfied. (This is in line with our general theorem of Chapter II). But they <u>can</u> be satisfied, and in infinitely many ways, if we permit negative masses. The restriction now is that m_1 and m_3 have the same sign, opposite to m_2.

We shall now prove, using a method whose basic idea is due to Hoffmann [21] that (3.69) has no solution, for non-vanishing values of the masses, when $n = 4$.

Define

$$\alpha_{ij} = \begin{cases} \dfrac{1}{m_i m_j} \; \nu_{ij}^{(o)} & (j < i) \\ 0 & (j = i) \\ -\dfrac{1}{m_i m_j} \; \nu_{ij}^{(o)} & (j > i) \end{cases} \tag{3.71}$$

then the elementary flatness condition (3.69) can be written as

$$m_i \sum_{j=1}^{n} \alpha_{ij} m_j = 0 \; . \tag{3.72}$$

A <u>necessary</u> condition that (3.72) be satisfied for non-vanishing values of the masses is

$$\det(\alpha_{ij}) = 0 \; . \tag{3.73}$$

This condition is trivially satisfied if n is odd, for it is well known that the determinant of an antisymmetric matrix vanishes if it is of odd order. If the determinant is of even order, it can be written as a perfect square:

$$\det(\alpha_{ij}) = \begin{cases} 0, & n \text{ odd} \\ \left[\frac{1}{2^m m!} \epsilon^{i_1 \ldots i_n} \alpha_{i_1 i_2} \alpha_{i_3 i_4} \ldots \alpha_{i_{n-1} i_n}\right]^2 & (n=2m) \end{cases} \tag{3.74}$$

When $n = 4$, the necessary condition for the elementary flatness, (3.73), therefore becomes

$$\alpha_{12}\,\alpha_{34} - \alpha_{13}\,\alpha_{24} + \alpha_{14}\,\alpha_{23} = 0 \ . \tag{3.75}$$

From (3.68) and (3.71) we have

$$\frac{1}{2}(\alpha_{13}-\alpha_{12}) = \frac{1}{b_1 b_3} \log\left[1 + \frac{b_1 b_3}{(a_1-a_3)^2-\frac{1}{4}(b_1+b_3)^2}\right]$$

$$+ \frac{1}{b_1 b_2} \log\left[1 - \frac{b_1 b_2}{(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2}\right]$$

$$< \frac{1}{(a_1-a_3)^2-\frac{1}{4}(b_1+b_3)^2} - \frac{1}{(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2}$$

$$= \frac{(a_1-a_2)^2-(a_1-a_3)^2+\frac{1}{4}(b_1+b_3)^2-\frac{1}{4}(b_1-b_2)^2}{[(a_1-a_3)^2-\frac{1}{4}(b_1+b_3)^2][(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2]}$$

$$= \frac{-(a_3-a_2)(a_2+a_3-2a_1)+\frac{1}{2}(b_2+b_3)(b_1+\frac{b_3-b_2}{2})}{[(a_1-a_3)^2-\frac{1}{4}(b_1+b_3)^2][(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2]}$$

$$< 0 \ .$$

where we have used the inequalities

$$\log(1+x) < x \qquad (x > -1)$$

$$a_i - a_j > \tfrac{1}{2}(b_i + b_j) \qquad (i > j)$$

the second of which follows from (3.67). Similarly, we can show that

$$\alpha_{24} - \alpha_{34} < 0 \quad .$$

It follows that

$$\alpha_{12}\,\alpha_{34} - \alpha_{13}\,\alpha_{24} + \alpha_{14}\,\alpha_{23} > 0$$

and hence (3.75) cannot be satisfied. We have shown that no static solution is possible for $n = 4$.

The results we have obtained lend support to a conjecture we shall call Hoffmann's conjecture:

Consider an axially symmetric configuration in which n bodies with masses of both signs are arranged sequentially along the z-axis. If n is even the configuration cannot be in equilibrium; if n is odd, there are an infinite number of equilibrium configurations.

We now drop the sequential condition (3.67). This admits configurations of the type considered by Marder [20]. The simplest situation of this type enables us to reinterpret the two-body solution given in § 3.6.

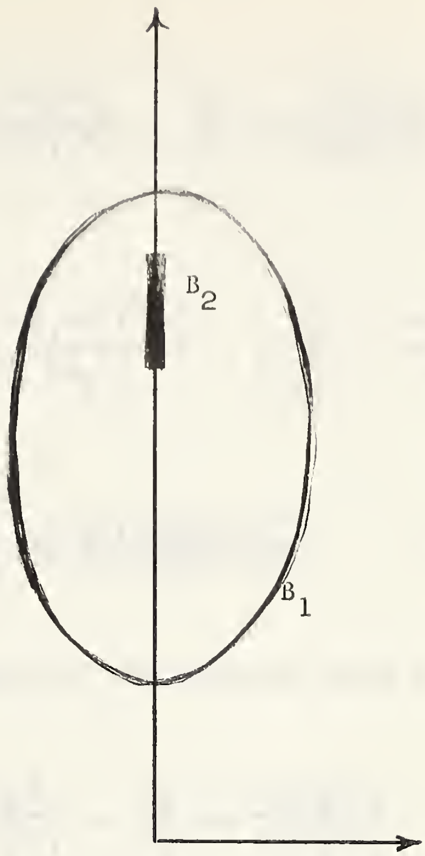

Figure 4.

(Body B_2 inside Body B_1)

In the Euclidean map let the first body B_1 be thin hollow spheroidal shell with mass m_1 and foci at $a_1 \pm \frac{1}{2}b_1$. We assume the equation of the shell to be $\rho_1 + \rho_1' = c_1$ where $c_1 > b_1$. Let the rod B_2 extending from $(a_2 - \frac{1}{2}b_2)$ to $(a_2 + \frac{1}{2}b_2)$ along the axis representing a second body enclosed within the shell.

For the field exterior to the shell, we have the two-body solution of § 3.6:

$$(3.76)\quad \begin{cases} \lambda = \dfrac{m_1}{b_i} \log \dfrac{\rho_1+\rho_1'-b_1}{\rho_1+\rho_1'+b_1} + \dfrac{m_2}{b_2} \log \dfrac{\rho_2+\rho_2'-b_2}{\rho_2+\rho_2'+b_2} \\ \\ \nu = 2\dfrac{m_1^2}{b_1^2} \log \dfrac{(\rho_1+\rho_1')^2-b_1^2}{4\rho_1\rho_1'} + 2\dfrac{m_2^2}{b_2^2} \log \dfrac{(\rho_2+\rho_2')^2-b_2^2}{4\rho_2\rho_2'} + \end{cases}$$

$$+ 2\frac{m_1m_2}{b_1b_2} \log \frac{E(1',2)E(1,2')}{E(1,2)E(1',2')} \qquad (\rho_1+\rho_1' > c_1) .$$

The field in the interior of the shell is given by

$$(3.77)\quad \begin{cases} \lambda = \dfrac{m_1}{b_1} \log \dfrac{c_1-b_1}{c_1+b_1} + \dfrac{m_2}{b_2} \log \dfrac{\rho_2+\rho_2'-b_2}{\rho_2+\rho_2'+b_2} \\ \\ \nu = 2\dfrac{m_2^2}{b_2^2} \log \dfrac{(\rho_2+\rho_2')-b_2^2}{4\rho_2\rho_2'} \qquad (\rho_1+\rho_1' < c_1) . \end{cases}$$

We turn to the elementary flatness condition, $\nu = 0$ when $r = 0$. It is easy to check that this is satisfied on segments of the z-axis exterior to the shell, for which $\rho_i = z_i$, $\rho_i' = z_i'$, $r = 0$ (upper segment) or $\rho_i = -z_i$, $\rho_i' = -z_i'$, $r = 0$ (lower segment). For segments of the z-axis interior to shell, (3.77) shows that $\nu = 0$ except on the section actually occupies by the rod: $\rho_2 + \rho_2' = b_2$.

In (3.76), (3.77) we thus have a two-body solution satisfying elementary flatness and in which the masses of the two bodies are quite arbitrary.

The solution given above differs from that originally given by Marder. Marder considered a thick hollow shell, and used a Curzon-type solution, which is more specialized than the one given above. However, both solutions bring out the essential features equally well.

CHAPTER IV

STATIONARY AXIALLY SYMMETRIC VACUUM FIELDS

§ 4.1 Introduction.

In Chapter II, we defined what is generally meant by a stationary, axially symmetric space-time manifold (page17). We showed that co-ordinates x^1, x^2, $x^3 = \varphi$, $x^4 = t$ can be introduced in terms of which the metric form reduces to

$$(4.1) \qquad \Phi = e^{2\psi}(dx_1^2 + dx_2^2) + \ell d\varphi^2 + 2m d\varphi dt - f\, dt^2 \quad ,$$

where all metric coefficients are functions of x_1, x_2 only. The form of this metric is clearly unchanged by any conformal transformation of x_1, x_2 . The Einstein field equations associated with (4.1) have been studied by Andress [30], Lewis [34], Van Stockum [35], Papapetrou [36] and Tiwari and Misra [37]. To Lewis is due the important discovery that the vacuum field equations can be significantly simplified by the introduction of so-called 'canonical co-ordinates" (x_1, x_2). (These co-ordinates are similar to the canonical coordinates of Weyl for the static case (page 24), and in fact reduce to them if $m = 0$.) Even with this simplification, solving the vacuum field equations still entails finding solutions of two coupled non-linear partial differential equations in two unknown functions. To date, the only result obtained along these lines which seems to have even a remote physical significance is the one-dimensional solution

found by Lewis and discussed at some length by Van Stockum. This presumably represents the external field of a rotating infinite cylinder. Van Stockum has also given a corresponding interior solution, representing a rigidly rotating cylinder of dust held in dynamical equilibrium by gravitation and centrifugal force. Other solutions have been given by Misra, but their interest appears to be mainly mathematical.

A few months ago, R. P. Kerr [38] reported briefly on a stationary axially symmetric vacuum field which he had obtained along quite different lines, and which appears to represent the external field of a rotating sphere or ring.

The plan of this chapter is as follows.

In § 4.2, we derive the vacuum equations for the metric (4.1), and introduce the canonical co-ordinates of Lewis. In § 4.3, we reduce the problem of solving the vacuum equations to the solution of two coupled partial differential equations. Some of the developments of this section are new. In § 4.4 we give a new and very much simplified derivation of Lewis's solution for the external field of a rotating cylinder. The final section, § 4.5, is devoted to a very brief discussion of Kerr's solution.

§ 4.2 Field equations. Canonical co-ordinates.

Consider the metric form

$$\Phi = g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta} + g_{AB}\, dx^{A}\, dx^{B} \tag{4.2}$$

where

$$g_{\alpha\beta} \equiv g_{\alpha\beta}(x_1, x_2) \ , \quad g_{AB} = g_{AB}(x_1, x_2) \tag{4.3}$$

(α, β, γ etc. = 1 or 2 ; A, B, C etc. = 3 or 4), which is a slight generalization of (4.1). We denote by a stroke covariant differentiation with respect to the two-dimensional metric

$$g_{\alpha\beta} \, dx^{\alpha} \, dx^{\beta} \tag{4.4}$$

and a comma indicates partial differentiation, e.g.

$$g_{AB,\mu} = \partial_{\mu} \, g_{AB} \ .$$

Note that g_{AB} is a <u>scalar</u>, and $g_{AB,\mu}$ a <u>vector</u> with respect to transformations in the 2-space of the metric (4.4).

The components of the Ricci tensor corresponding to the line-element (4.2) have been given in an elegant and concise form by Van Stockum [35], whose presentation we follow here.

Write

$$D^2 = - \det \| g_{AB} \| \tag{4.5}$$

$$\| g^{AB} \| = \| g_{AB} \|^{-1} \ , \quad \| g^{\alpha\beta} \| = \| g_{\alpha\beta} \|^{-1} \ . \tag{4.6}$$

Then the Christoffel symbols for the line-element (4.2) are as follows:

$$
(4.7)\qquad \begin{cases} \Gamma^A_{BC} = 0\ , \quad \Gamma^A_{\alpha\beta} = 0\ , \quad \Gamma^\alpha_{\beta A} = 0 \\[2ex] \Gamma^A_{B\alpha} = \tfrac{1}{2} g^{AC} g_{BC,\alpha}, \quad \Gamma^\alpha_{AB} = -\tfrac{1}{2} g^{\alpha\beta} g_{AB,\beta} \end{cases}
$$

For the components of the Ricci tensor, we then find

$$
(4.8)\qquad \begin{cases} R^\alpha_\beta = K^\alpha_\beta + D^{-1}(D_{|\beta\mu} - D_{,\beta} D_{,\mu}) g^{\alpha\mu} \\[2ex] R^A_B = \tfrac{1}{2} D^{-1} (D g^{AC} g_{BC,\alpha})_{|\beta} g^{\alpha\beta} \end{cases}
$$

where K^α_β is the Ricci tensor of 2-space (4.4).

We now specialize to the particular metric (4.1), i.e. we take

$$
(4.9)\qquad \begin{cases} \| g_{AB} \| = \left\| \begin{matrix} \ell & m \\ m & -f \end{matrix} \right\| , \qquad D^2 = \ell f + m^2 \\[2ex] \| g_{\alpha\beta} \| = \left\| \begin{matrix} e^{2\psi} & o \\ o & e^{2\psi} \end{matrix} \right\| \ . \end{cases}
$$

Since space-time is normal-hyperbolic, it is necessary that D^2 defined by (4.9) be positive, as we have anticipated by the notation. Equations (4.8) now reduce to

$$
(4.10)\qquad \sqrt{-g}\, R^\alpha_\beta = D(\Delta\psi)\,\delta^\alpha_\beta + D_{,\alpha|\beta} - \tfrac{1}{4} D^{-1} (\ell_{,\alpha} f_{,\beta} + \ell_{,\beta} f_{,\alpha} + 2m_{,\alpha} m_{,\beta})
$$

$$
(4.11)\qquad \sqrt{-g}\, R^3_3 = \tfrac{1}{2}\partial_\alpha \left[\frac{f\ell_{,\alpha} + m\, m_{,\alpha}}{D} \right] \ .
$$

$$\sqrt{-g}\, R^{3}_{4} = \tfrac{1}{2}\partial_\alpha \left[\frac{fm_{,\alpha} - mf_{,\alpha}}{D} \right] \tag{4.12}$$

$$\sqrt{-g}\, R^{4}_{3} = \tfrac{1}{2}\partial_\alpha \left[\frac{m\ell_{,\alpha} - \ell m_{,\alpha}}{D} \right] \tag{4.13}$$

$$\sqrt{-g}\, R^{4}_{4} = \tfrac{1}{2}\partial_\alpha \left[\frac{\ell f_{,\alpha} + mm_{,\alpha}}{D} \right] \tag{4.14}$$

where $\Delta\psi = \psi_{,\alpha\alpha} = \dfrac{\partial^2\psi}{\partial x_1^2} + \dfrac{\partial^2\psi}{\partial x_2^2}$.

We proceed to explore the consequences of the vacuum field equations

$$R^{m}_{n} = 0 \ . \tag{4.15}$$

From (4.11) and (4.14) we have

$$0 = \sqrt{-g}(R^{3}_{3} + R^{4}_{4}) = \tfrac{1}{2}\partial_\alpha \left[\frac{(\ell f + m^2)_{,\alpha}}{D} \right] \tag{4.16}$$

$$= \Delta D \ .$$

Hence $r = D(x_1, x_2)$ <u>is a harmonic function in vacuo</u> (more generally: in a medium whose energy tensor satisfies $T^1_1 + T^2_2 = 0$) . Let $z(x_1, x_2)$ be the conjugate harmonic function, so that the analytic transformation $(x_1, x_2) \rightarrow (r, z)$:

$$r + iz = f(x_1 + ix_2)$$

is conformal. This transformation leaves the form of the metric (4.1), and hence the form of equations (4.10) to (4.14), unchanged.

We can thus rewrite our previous equations in terms of the "canonical co-ordinates" (r,z). We have

$$(4.17) \qquad \Phi = e^{2\psi(r,z)}\,(dr^2 + dz^2) + \ell(r,z)\,d\varphi^2 + 2m(r,z)\,d\varphi dt - f(r,z)\,dt^2 \;,$$

with

$$(4.18) \qquad f\ell + m^2 = r^2 \;.$$

The Ricci tensor for this metric is given by (4.10) to (4.14), with $D = r$. Equation (4.16) is now an identity.

The introduction of the Lewis canonical co-ordinates r,z and the simplifications attendant upon their introduction, are always possible in vacuo or in a medium whose energy tensor satisfies

$$T^1_1 + T^2_2 = 0 \;,$$

since

$$\begin{aligned}(T^1_1 + T^2_2) &= -K[R^1_1 + R^2_2 - \tfrac{1}{2}\delta^1_1 R - \tfrac{1}{2}\delta^2_2 R] \\ &= K(R^3_3 + R^4_4) \\ &= \frac{K}{\sqrt{-g}}\,\Delta D\end{aligned}$$

by (4.16). We observe from (4.17) and (4.18) that, when $m = 0$, r, z reduce to the canonical co-ordinates of Weyl for the static case [c.f. Chapter II, page 24].

We proceed to unpack the remaining equation of (4.15). From (4.11), and (4.14),

$$\sqrt{-g}(R^3_3 - R^4_4) = \tfrac{1}{2}\partial_\alpha\left(\frac{f\ell_{,\alpha} - \ell f_{,\alpha}}{r}\right) = 0$$

which yields

$$(4.19) \qquad f\Delta^*\ell = \ell\Delta^* f$$

where (for any function $\psi(r,z)$)

$$(4.20) \qquad \Delta^*\psi \equiv \Delta\psi - \frac{1}{r}\psi_{,1} = \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} .$$

The equations $R^3_4 = 0$, $R^4_3 = 0$ yield

$$(4.21) \qquad m\Delta^*\ell = \ell\Delta^* m ,$$

$$(4.22) \qquad f\Delta^* m = m\Delta^* f .$$

Any one of the three equations (4.19), (4.21), (4.22) is a consequence of the other two.

Operating with Δ^* on the relation $\ell f + m^2 = r^2$, and noting $\Delta^* r^2 = 0$, we find

$$(4.23) \qquad \ell\Delta^* f + f\Delta^*\ell + 2\nabla f \cdot \nabla\ell + 2m\Delta^* m + 2(\nabla m)^2 = 0$$

where, for any two functions $\varphi(r,z)$, $\psi(r,z)$, we define

$$(4.24) \qquad \nabla\varphi \cdot \nabla\psi = \frac{\partial\varphi}{\partial r} \cdot \frac{\partial\psi}{\partial r} + \frac{\partial\varphi}{\partial z}\,\frac{\partial\psi}{\partial z} \quad .$$

From (4.19) and (4.23)

$$\begin{aligned} f\,\Delta^*\,\ell &= \tfrac{1}{2}(f\,\Delta^*\,\ell + \ell\,\Delta^*\,f) \\ &= -\,m\,\Delta^*\,m - \nabla f \cdot \nabla\ell - (\nabla m)^2 \\ &= -\frac{m^2}{\ell}\,\Delta^*\,\ell - \nabla f \cdot \nabla\ell - (\nabla m)^2 \quad \text{by (4.21)} \ . \end{aligned}$$

Hence, using $f\ell + m^2 = r^2$,

$$(4.25) \qquad r^2\,\Delta^*\,\ell + \ell[\nabla f \cdot \nabla\ell + (\nabla m)^2] = 0 \ .$$

From (4.19) and (4.21), it then follows that

$$(4.26) \qquad r^2\,\Delta^*\,f + f[\nabla f \cdot \nabla\ell + (\nabla m)^2] = 0$$

$$(4.27) \qquad r^2\,\Delta^*\,m + m[\nabla f \cdot \nabla\ell + (\nabla m)^2] = 0 \ .$$

We turn to the equations $R^\alpha_\beta = 0$ $(\alpha,\beta = 1 \text{ or } 2)$. We have, since $D = r$

$$D_{,1} = 1 \ , \quad D_{,2} = 0$$

$$D_{,\alpha|\beta} = -\,\Gamma^1_{\alpha\beta} \quad .$$

The equations $R^1_2 = 0$ give

$$D_{,1|2} - \frac{1}{4r}(\ell_{,1}\, f_{,2} + \ell_{,2}\, f_{,1} + 2m_{,1}\, m_{,2}) = 0$$

which can be written

$$\frac{\partial\psi}{\partial z} = -\frac{1}{4r}\left(\frac{\partial\ell}{\partial r}\cdot\frac{\partial f}{\partial z} + \frac{\partial\ell}{\partial z}\cdot\frac{\partial f}{\partial r} + 2\frac{\partial m}{\partial r}\cdot\frac{\partial m}{\partial z}\right) . \tag{4.28}$$

The equations $R^1_1 - R^2_2 = 0$ lead to

$$D_{,1|1} - D_{,2|2} - \frac{1}{4r}(2\ell_{,1}\, f_{,1} - 2\ell_{,2}\, f_{,2} + 2m^2_{,1} - 2m^2_{,2}) = 0$$

or

$$\frac{\partial\psi}{\partial r} = -\frac{1}{4r}\left[\frac{\partial\ell}{\partial r}\cdot\frac{\partial f}{\partial r} - \frac{\partial\ell}{\partial z}\frac{\partial f}{\partial z} + \left(\frac{\partial m}{\partial r}\right)^2 - \left(\frac{\partial m}{\partial z}\right)^2\right] . \tag{4.29}$$

The integrability condition of (4.28) and (4.29) is

$$\frac{\partial\ell}{\partial z}\,\Delta^* f + \frac{\partial f}{\partial z}\,\Delta^*\ell + 2\frac{\partial m}{\partial z}\,\Delta^* m = 0 \tag{4.30}$$

and is identically satisfied because of (4.18), (4.21) and (4.22).

It is routine to check that the equations $R^1_1 = R^2_2 = 0$ are a consequence of (4.28) and (4.29), so that <u>the system (4.25) to (4.29), with $f\ell + m^2 = r^2$, is fully equivalent to the vacuum equations $R^m_n = 0$.</u>

§ 4.3 Reduction of the vacuum field equations.

We have already pointed out that, of the three equations (4.25), (4.26) and (4.27) for the three functions ℓ, f and m , one is redundant. This can be regarded as a consequence of the identity

$$\ell f + m^2 = r^2 . \tag{4.31}$$

We shall now make this redundancy more apparent by reducing the three functions ℓ, f, m to effectively two.

First let us write

$$\ell = ru \ , \ f = rv \ , \ m = rw \tag{4.32}$$

so that (4.31) becomes

$$uv + w^2 = 1 \ . \tag{4.33}$$ We have, for any function $\psi(r,z)$

$$\Delta^*(r,\psi) = r\,\Delta^*\,\psi + \psi\,\Delta^*\,r + 2\nabla\psi\cdot\nabla r$$

$$= r\,\Delta^*\,\psi - \psi\,\frac{1}{r} + 2\,\frac{\partial\psi}{\partial r}$$

$$= r\,\nabla^2\,\psi - \frac{\psi}{r}$$

where

$$\nabla^2\,\psi\,(r,z) = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} \ ; \tag{4.34}$$

$\nabla^2\psi$ is formally the Lapacian for cylindrical co-ordinates r, z, φ in Euclidean space. Equations (4.25), (4.26) and (4.27) now become

(4.35) $$\nabla^2 u + Pu = 0$$

(4.36) $$\nabla^2 v + Pv = 0$$

(4.37) $$\nabla^2 w + Pw = 0$$

where

(4.38) $$rP = -1 + \nabla f \cdot \nabla \ell + (\nabla m)^2$$

$$= r^2[\nabla u \cdot \nabla v + (\nabla w)^2] \quad .$$

We have made use of (4.33) in simplifying the expression for P .

We now replace ℓ, f, m by two algebraically independent functions $\mu(r,z)$, $\theta(r,z)$ in such a way that (4.33) is identically satisfied.

<u>Case I : Assume $m^2 \leq r^2$, i.e. $w^2 \leq 1$,</u>

Let

(4.39) $$u = e^{-\mu} \sin\theta \quad , \quad v = e^{\mu} \sin\theta \ , \quad w = \cos\theta \ .$$

This satisfies (4.33) for all μ and θ .

We then obtain from (4.38)

(4.40) $$P = r[(\nabla\theta)^2 - \sin^2\theta\,(\nabla\mu)^2] \quad .$$

Equations (4.35), (4.36) and (4.37) reduce to

$$\sin\theta[\nabla^2\mu + 2(\cot\theta)\nabla\mu \cdot \nabla\theta] - \cos\theta[\nabla^2\theta + \sin\theta\cos\theta \ (\nabla\mu)^2] = 0 \ ,$$

$$\sin\theta[\nabla^2\mu + 2(\cot\theta)\nabla\mu \cdot \nabla\theta] + \cos\theta[\nabla^2\theta + \sin\theta\cos\theta \ (\nabla\mu)^2] = 0$$

(4.41) $$\nabla^2\theta + \sin\theta\cos\theta\,(\nabla\mu)^2 = 0 .$$

Thus, (4.35), (4.36) and (4.37) are equivalent to the <u>two</u> equations (4.41) and

(4.42) $$\nabla^2\mu + 2\cot\theta(\nabla\mu\cdot\nabla\theta) = 0 .$$

Equations (4.29) and (4.30) can be written, in view of (4.32) and (4.40),

(4.43) $$\frac{\partial\psi}{\partial r} = -\frac{1}{4r} - \frac{r}{4}\left[\theta^2_{,1} - \theta^2_{,2} - \sin^2\theta\,(\mu^2_{,1} - \mu^2_{,2})\right]$$

(4.44) $$\frac{\partial\psi}{\partial z} = \frac{r}{2}\left(\sin^2\theta\cdot\mu_{,1}\mu_{,2} - \theta_{,1}\theta_{,2}\right) .$$

<u>We thus have the following prescription for obtaining stationary axially symmetric vacuum fields with metric (4.17), in the case where $m^2 \le r^2$. Solve the coupled equations (4.41) and (4.42) for $\mu(r,z)$ and $\theta(r,z)$. Integrate (4.43) and (4.44) to get $\psi(r,z)$. The coefficients ℓ, m, f in the metric (4.17) are then given by</u>

(4.45) $$\ell = r\,e^{-\mu}\sin\theta\ ,\quad f = r\,e^{\mu}\sin\theta\ ,\quad m = r\cos\theta\ .$$

Unfortunately, the system (4.41), (4.42) is in general intractable.

Two special cases are of interest.

(i) If we set $\theta = \frac{\pi}{2}$, we come back to the static case,

$$m = 0 .$$

Equations (4.41) and (4.42) reduce to

(4.46) $$\nabla^2\mu = 0 ,$$

while (4.43) and (4.44) become

(4.47) $$\frac{\partial\psi}{\partial r} = -\frac{1}{4r} + \frac{r}{4}(\mu^2_{,1} - \mu^2_{,2})$$

(4.48) $$\frac{\partial\psi}{\partial z} = \frac{r}{2}\mu_{,1}\mu_{,2} .$$

These are precisely the Weyl-Levi-Civita equations (2.47) and (2.48) of Chapter II, if we set

$$re^{\mu} = e^{2\lambda} , \quad \psi = \nu - \lambda .$$

(ii) Set $\mu = \log K$ (constant). Then (4.41) and (4.42) reduce to

(4.49) $$\nabla^2\theta = 0$$

and (4.43), (4.44) become

(4.50) $$\frac{\partial\psi}{\partial r} = -\frac{1}{4r} - \frac{r}{4}[\theta^2_{,1} - \theta^2_{,2}]$$

(4.51) $$\frac{\partial\psi}{\partial z} = -\frac{r}{2}\theta_{,1}\theta_{,2} .$$

The corresponding metric is

(4.52) $$\Phi = e^{2\psi}(dr^2 + dz^2) + r\sin\theta(Kd\varphi^2 - \frac{1}{k}dt^2) + 2r\cos\theta\, d\varphi\, dt .$$

Since (4.49) is formally Laplace's equation in cylindrical co-ordinates, a large class of vacuum manifolds of the type (4.52) can immediately be found.

This class of solution was first isolated by Tiwari and Misra [37].

Case II: We now suppose $m^2 \geq r^2$, i.e. $w^2 \geq 1$, corresponding to a field in which rotational effects are very strong.

Equation (4.33) is automatically satisfied if we set

$$u = e^{-\mu}\sinh\theta, \quad v = -e^{\mu}\sinh\theta, \quad w = \cosh\theta \quad . \tag{4.53}$$

We observe that this makes the coefficient of dt^2 positive, or else the co-efficient of $d\varphi^2$ negative. However, the normal hyperbolic character of the metric is left inviolate, since it is entirely determined by the positiveness of $\ell f + m^2$.

Equation (4.37) yields

$$P = -r[(\nabla\theta)^2 - \sinh^2\theta(\nabla\mu)^2] \quad . \tag{4.54}$$

Equations (4.35), (4.36) and (4.37) become

$$\sinh\theta[\nabla^2\mu + 2\coth\theta\nabla\mu\cdot\nabla\theta]$$

$$-\cosh\theta[\nabla^2\theta + \sinh\theta\cosh\theta\,(\nabla\mu)^2] = 0 \,,$$

$$\sinh\theta[\nabla^2\mu + 2\coth\theta\nabla\mu\cdot\nabla\theta]$$

$$+\cosh\theta[\nabla^2\theta + \sinh\theta\cosh\theta\,(\nabla\mu)^2] = 0 \,,$$

$$\nabla^2\theta + \sinh\theta\cosh\theta\,(\nabla\mu)^2 = 0 \,, \tag{4.55}$$

and are therefore equivalent to the two equations (4.55) and

$$\nabla^2\mu + 2\coth\theta\nabla\mu\cdot\nabla\theta = 0 \quad . \tag{4.56}$$

Equations (4.28) and (4.29) can be written, by virtue of (4.32) and (4.53),

(4.57) $$\frac{\partial\psi}{\partial r} = -\frac{1}{4r} + \frac{r}{4}\left[\theta_{,1}^{2} - \theta_{,2}^{2} - \sinh^{2}\theta(\mu_{,1}^{2} - \mu_{,2}^{2})\right]$$

(4.58) $$\frac{\partial\psi}{\partial z} = -\frac{r}{2}(\sinh^{2}\theta\ \mu_{,1}\mu_{,2} - \theta_{,1}\theta_{,2}) \quad .$$

In summary, stationary axially symmetric vacuum fields (4.17) may be obtained by solving the system (4.55) to (4.58), and then setting

(4.59) $$f = re^{-\mu}\sinh\theta \ , \quad \ell = -re^{\mu}\sinh\theta \ , \quad m = r\cosh\theta \ .$$

A special, tractable class of solutions may again be obtained by setting $\mu = \log k$. The metric (4.17) then reduces to

(4.60) $$\Phi = e^{2\psi}(dr^{2}+dz^{2}) - r\sinh\theta\left(\frac{1}{K}\,d\varphi^{2} + K dt^{2}\right) + 2r\cosh\theta\ d\varphi\ dt \ ,$$

with $\theta(r,z)$ and $\psi(r,z)$ determinable from

(4.61) $$\nabla^{2}\theta = 0$$

(4.62) $$\frac{\partial\psi}{\partial r} = -\frac{1}{4r} + \frac{r}{4}(\theta_{,1}^{2} - \theta_{,2}^{2})$$

(4.63) $$\frac{\partial\psi}{\partial z} = \frac{r}{2}\,\theta_{,1}\,\theta_{,2} \ .$$

The above reduction, and the isolation of the special class of solutions (4.60), have not previously been given in the literature.

§ 4.4 Lewis's solution. A new derivation.

In the special classes of solutions (4.60), (4.52) which we isolated in the previous section, let us assume θ to be a function of r only. To satisfy $\nabla^2\theta = 0$, we have to take

$$\theta = c_1 \log r + \log c_2 \quad . \tag{4.64}$$

For the case $m^2 \geq r^2$, (4.62) and (4.63) then yield

$$\psi = \tfrac{1}{4}(c_1^2 - 1) \log r + \text{const.} \;;$$

while for the case $m^2 \leq r^2$, we have from (4.50) and (4.51)

$$\psi = -\tfrac{1}{4}(c_1^2 + 1) \log r + \text{constant.}$$

We thus arrive at the two stationary, cylindrical symmetric vacuum metrics.

$$\Phi = Ar^{\frac{1}{2}(c_1^2-1)}(dr^2+dz^2) - \tfrac{1}{2}\left(c_2 r^{1+c_1} - \frac{1}{c_2} r^{1-c_1}\right) \left(\frac{1}{K}\, d\varphi^2 + K\, dt^2\right) + \left(c_2\, r^{1+c_1} + \frac{1}{c_2} r^{1-c_1}\right) d\varphi\, dt \,, \tag{4.65}$$

$$\Phi = Ar^{-\frac{1}{2}(c_1^2+1)}(dr^2+dz^2) + r \sin(c_1 \log r + c_2)\left(K\, d\varphi^2 - \frac{1}{K}\, dt^2\right) + 2r \cos(c_1 \log r + c_2)\, d\varphi\, dt \,. \tag{4.66}$$

The metric (4.65) is the solution originally given by Lewis [34], here obtained by a new route. The metric (4.66) was given by Van Stockum

[35] who has interpreted both solutions as the external fields of rotating infinite cylinders, and has shown how they can be joined to suitable interior solutions. We shall not pursue these questions further here.

§ 4.5 Kerr's solution.

The remarkable and important solution recently obtained by R. P. Kerr [38] was derived by a method which bears no resemblance to that developed for stationary axially symmetric fields in §§ 4.2 and 4.3. It may, in fact be fairly described as an accidental and very fortunate by-product of an investigation not directly connected with the problem of axially symmetric fields. Details of Kerr's calculations have not yet been published, and it will be impossible to give them here. But, before presenting his explicit solution, I shall attempt to indicate in a qualitative way the kind of problem he was concerned with.

An algebraically special vacuum field is a space-time free of matter and electromagnetic radiation $(R_{ij} = 0)$ in which there exists at least one null vector field K^{α} satisfying

(4.67) $$R_{\alpha\beta\gamma[\delta} K_{\epsilon]} K^{\beta} K^{\gamma} = 0$$

Goldberg and Sachs [57] have proved that: a vacuum field is algebraically special if and only if it contains a shear-free null geodesic congruence.

A null geodesic congruence is a family of null geodesics exactly one of which passes through each point of the manifold. The congruence is

said to be <u>shear-free</u> if the field of null tangent vectors to the geodesics satisfies

(4.68) $$(K_{\alpha|\beta} + K_{\beta|\alpha})\, K^{\alpha|\beta} = (K^{\alpha}_{|\alpha})^2 \quad .$$

The geometrical significance of this condition has been explained by Sachs [58]. According to the above theorem of Goldberg and Sachs the null vector field satisfying (4.68) in an empty space-time also satisfies (4.67).

In studying algebraically special fields, it is natural to focus attention on the null vector field K_α and its variation along the null geodesic rays. This is accomplished by choosing K_α as one of the vectors of an orthogonal tetrad. Let m_α be a second null vector, satisfying

$$m_\alpha m^\alpha = 0 \quad , \quad m_\alpha K^\alpha = 1 \, .$$

In the space-like 2-flat orthogonal to the 2-flat containing m_α, K_α choose a pair of orthogonal vectors Y_α , Z_α , which thus satisfy

$$Y_\alpha Z^\alpha = 0 \, , \quad Y_\alpha Y^\alpha = 1 \, , \quad Z_\alpha Z^\alpha = 1$$

$$Y_\alpha m^\alpha = Z_\alpha m^\alpha = Y_\alpha K^\alpha = Z_\alpha K^\alpha = 0 \quad .$$

It is then easy to see that we can express the metric tensor in the form

(4.69) $$g_{\alpha\beta} = m_\alpha K_\beta + m_\beta K_\alpha + Y_\alpha Y_\beta + Z_\alpha Z_\beta \quad ,$$

for we merely have to check that $g_{\alpha\beta} A^{\beta} = A_{\alpha}$ for all vectors A^{β} when $g_{\alpha\beta}$ is defined by (4.69). (Since the vectors m^{β}, K^{β}, Y^{β}, Z^{β} span the four-dimensional manifold, it is enough to verify that $g_{\alpha\beta} m^{\beta} = m_{\alpha}$ and three similar relations.)

It is customary to introduce the complex null vectors

$$t^{\alpha} = \frac{1}{\sqrt{2}} (Y^{\alpha} + iZ^{\alpha}) \ , \quad \bar{t}^{\alpha} = \frac{1}{\sqrt{2}} (Y^{\alpha} - iZ^{\alpha})$$

so that

$$t_{\alpha} t^{\alpha} = \bar{t}_{\alpha} \bar{t}^{\alpha} = 0 \ , \quad t_{\alpha} \bar{t}^{\alpha} = 1$$

and (4.69) can be written

$$g_{\alpha\beta} = m_{\alpha} K_{\beta} + m_{\beta} K_{\alpha} + t_{\alpha} \bar{t}_{\beta} + \bar{t}_{\alpha} t_{\beta} \ .$$

The method used by Kerr and others in treating algebraically special vacuum fields is to re-write the field equations $R_{ij} = 0$ and the condition (4.67) in terms of the tetrad vectors K, m, t, $\bar{t}$. I shall not attempt to go into the complicated details of this procedure.

In the course of his investigation of algebraically special fields Kerr found the following particular vacuum field:

$$(4.70) \qquad \Phi = (R^2+a^2\cos^2\theta)(d\theta^2+\sin^2\theta\, d\varphi^2)+2(du+a\sin^2\theta d\varphi)(dR+a\sin^2\theta\, d\varphi)$$

$$- \left(1 - \frac{2mR}{R^2+a^2\cos^2\theta}\right)(du + a\sin^2\theta\, d\varphi)^2 \ .$$

It is clear that this line-element is axially symmetric (metric co-efficients independent of φ) and "stationary" in the generalized sense that the metric co-efficients are independent of u . Whether it is stationary in the narrower sense of Chapter II (page 17: invariance under simultaneous inversion of "time" and reversal of the sense of rotation) is not clear from the form (4.70). In fact, the entire relationship of Kerr's solution to the Lewis-type approach remains mysterious and would be an interesting topic for detailed investigation.

If in (4.70), we set the parameter a equal to zero, we obtain

(4.71) $$\Phi = R^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + 2\, du\, dR - (1-\frac{2m}{R})\, du^2 \quad .$$

This line-element represents an analytic extension of the Schwarzschild manifold [c.f. Chapter V] which resembles Finkelstein's well-known extension [52]. To convert (4.71) into its more familiar form, introduce a new co-ordinate T , defined for $R > 2m$ by

(4.72) $$u = R + T + 2m \log (\frac{R}{2m} - 1)$$

so that

(4.73) $$du = \frac{dR}{1-\frac{2m}{R}} + dT \quad .$$

The line-element (4.71) becomes

$$\Phi = \frac{dR^2}{1-\frac{2m}{R}} + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - (1-\frac{2m}{R})\, dT^2 \quad .$$

Thus, Kerr's solution includes Schwarzschild's as a particular case $(a=0)$.

Kerr has indicated another way in which the general solution (4.70) can be expressed in terms of asymptotically Galilean co-ordinates x,y,z,t . Make the transformation

$$(4.74) \qquad \begin{cases} x = (R\cos\varphi + a\sin\varphi)\sin\theta \\ y = (R\sin\varphi - a\cos\varphi)\sin\theta \\ z = R\cos\theta\ , \quad u = R + t\ . \end{cases}$$

We then obtain from (4.70),

$$(4.75) \qquad \Phi = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2mr^3}{r^4+a^2z^2}\, q^2$$

where q stands for a Pfaffian form defined by

$$(4.76) \qquad (r^2+a^2)rq = r^2(xdx+ydy) + ar(xdy-ydx) + (r^2+a^2)(zdz+rdt)$$

where the function r is defined by

$$(4.77) \qquad r^4 - (\rho^2-a^2)r^2 - a^2z^2 = 0\ , \quad \rho^2 = x^2+y^2+z^2\ .$$

For $r \gg a$ we have

$$r = \rho + 0(\tfrac{a}{\rho}) \qquad (\tfrac{\rho}{a} \to \infty)\ .$$

In this co-ordinate system the solution is regular everywhere except at $\rho = a\ ,\ z = 0$.

Kerr reports that he has compared the field (4.75) at large distances with known approximate results for the exterior field of a rotating sphere. There is agreement to the third order of the Einstein-Infeld-Hoffmann approximation if m is taken to be the mass and ma the angular momentum about the z-axis.

Very recently Newman and Janis reported to the January 1964 meeting of the American Physical Society in New York on a new interpretation of the Kerr solution. The abstract [39] of their (as yet unpublished) paper reads as follows:

> "It is shown, that, by means of a complex co-ordinate transformation performed on the monopole or Schwarzschild metric, one obtains a new metric (first discovered by Kerr). It has been suggested that this metric be interpreted as that arising from a spinning particle. We wish to suggest a more complicated interpretation: namely, that the metric arises from a ring of mass that is rotating about its axis of symmetry. The argument for this interpretation comes from 3 separate places:
>
> (1) the metric appears to have the appropriate multipole structure when analyzed in the manner discussed earlier;
>
> (2) in a covariantly defined flat space associated with the metric, the Riemann tensor has a circular singularity;
>
> (3) there exists a closely analogous solution of Maxwell's equations that is the field of a rotating ring of charge."

CHAPTER V

BONDI'S PROBLEM

§ 5.1 Introduction.

Under "normal" conditions, Newtonian theory is a remarkably close approximation to general relativity, apart from one or two minor points (advance of perihelion of Mercury etc.). Indeed, our discussion up to now has relied quite heavily on the analogy between the Newtonian and relativistic theories. On the debit side, the predictions of Einstein's theory under "normal" conditions can lead only to results which have been essentially familiar for 250 years. The most interesting applications of general relativity are to those "exotic" situations which classical theory is inadequate or powerless to deal with. It is largely for this reason that fields remote from present day experimental varification, such as gravitational waves and cosmology, are so heavily cultivated by research workers. Our object in this chapter is to deal with one such "exotic" application.

In an essay which was awarded the first Gravity Research Foundation Prize in 1957, Hermann Bondi [7] considered the following problem. Two free particles of masses $+m$, $-m$ are simultaneously started from rest. What happens?

The conclusions of a naive Newtonian analysis of this problem are rather startling. According to Newton's law of gravitation if $\vec{r}_1(t)$, $\vec{r}_2(t)$ are the position vectors of the negative and positive masses, and

if $\vec{F}_1, \vec{F}_2$ are the gravitational forces experienced by them, then we have

(5.1) $$\vec{F}_2 = -\vec{F}_1 = -\frac{(-m)(+m)}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1) .$$

Further, according to the second law of motion,

(5.2) $$\vec{F}_1 = (-m)\ddot{\vec{r}}_1 , \quad \vec{F}_2 = (+m)\ddot{\vec{r}}_2 .$$

We thus find

(5.3) $$\ddot{\vec{r}}_1 = \ddot{\vec{r}}_2 = \frac{m}{|\vec{r}_2-\vec{r}_1|^3}(\vec{r}_2-\vec{r}_1) .$$

<u>The negative and positive masses are equally accelerated in the SAME direction - the negative mass chases the positive mass</u>.

A little critical reflection shows, however, that the above argument has no secure foundation. We have assumed that the constants (+m), (-m) entering (5.1) are the same as those entering (5.2). The only support for this assumption is observational (the Eotvos experiment) and observation is (at least so far) confined to positive mass. <u>We have failed to distinguish between gravitational and inertial mass</u>. <u>Inertial mass</u> is the constant m_{inert} characteristic of a particle which measures its kinematic reaction to a given applied force:

(5.4) $$\vec{F} = m_{inert}\ddot{\vec{r}} .$$

Gravitational mass is the constant (analogous to charge in electrostatics)

entering Newton's law of gravitation. It is in fact possible to distinguish two kinds of gravitational mass. Active gravitational mass of a particle is the constant which measures the gravitational field generated by it, while its passive gravitational mass determines the force exerted on it in a given external gravitational field. Thus:

Gravitational force on particle 2 due to particle 1

$$= \frac{-(m_1)_{\text{active.grav}}\,(m_2)_{\text{pass.grav}}}{|\vec{r}_2 - \vec{r}_1|^3}\,(\vec{r}_2 - \vec{r}_1) \quad . \tag{5.5}$$

The situation is now much less clear-cut. It is, for instance, possible - to mention just one of 2^3 possibilities - that a particle with $m_{\text{active.grav}}$ negative has positive $m_{\text{pass.grav.}}$ and positive $m_{\text{inert.}}$. To speculate further along these lines seems profitless.

Let us now turn to consideration of Bondi's problem from a pseudo-relativistic point of view. Built into the foundations of Einstein's theory is the principle of equivalence. For our purposes, this may be stated in the following restricted form: All free particles at a given point of a gravitational field fall with the same acceleration. According to (5.4) and (5.5) this is only possible if the passive gravitational mass of every particle is equal to its inertial mass. We find

$$\ddot{\vec{r}}_2 = \frac{-(m_1)_{\text{act.grav.}}}{|\vec{r}_2-\vec{r}_1|^3}\,(\vec{r}_2 - \vec{r}_1) \quad . \tag{5.6}$$

The arbitrariness and uncertainty inherent in the Newtonian argument has been

eliminated by the principle of equivalence.

In considering the motion of a set of particles under their mutual gravitational influence, the only type of mass which needs to be considered is active gravitational mass. Particles with positive active gravitational mass attract all other particles; particles with negative active gravitational mass repel all other particles.

If we reconsider the problem of two particles with equal and opposite active gravitational masses from the present point of view, we are led to precisely the same conclusion as we were on the basis of our naive Newtonian argument. If the particles are started from rest, it appears that they will move as a rigid, uniformly accelerated unit - a "Bondi dipole" - with the acceleration directed from the negative to the positive mass.

The conclusion remains startling, and it would be desirable to have confirmation by the construction of an exact solution. Do Einstein's field equations admit a vacuum solution which represents a Bondi dipole and which is free of singularities except along the world-lines C_-, C_+ of the two opposite masses (Fig. 5)? To settle this question would appear to be a straightforward matter, since the field is static and axially symmetric when viewed from a co-moving uniformly accelerated frame. However, it is at present still an open question whether a global solution of this type exists. There is indeed no difficulty in obtaining a vacuum line-element whose domain of validity is the sector A O B of space-time bounded by the light-like asymptotes to C_-, C_+ ; what remains undecided

is whether it is possible to extend this line-element smoothly across the null-barrier into the rest of space-time.

Bondi was however able to prove the global existence of another type of solution involving <u>four</u> particles. This solution possesses reflexional symmetry and represents two identical Bondi dipoles moving in radially opposite directions. Bondi's result was purely an existence theorem. He did not give the solution explicitly. This is accomplished in § 5.3 of this chapter.

One of the basic objectives of this chapter is to disclose a remarkable connection between Bondi's extendability problem and a fundamental unsolved problem [48] of general relativity: the extendability of the Schwarzschild manifold of a particle in the presence of an external gravitational field. <u>Penetrating Bondi's null barrier is tantamount to penetrating the "Schwarzschild singularity" of a particle with infinite mass.</u>

The procedure is readily understood in terms of a semi-Newtonian analogue.

Consider a system of three collinear particles with masses m_o, m_1, m_2 . If one of the outer masses, $m_o > 0$, and its position are arbitrarily assigned, the masses and locations of the other two particles can be adjusted (in infinitely many ways) so that the system is in (unstable) equilibrium and static. We then have, necessarily, $m_1 < 0$, $m_2 > 0$. Now let the mass m_o and its distance from m_1, m_2 simultaneously become infinite in such a manner that its limiting field is homogeneous. Since a

homogeneous gravitational field can be transformed away by changing to a uniformly accelerated frame (c.f. Chapter III, p. 39) we expect to be left with a uniformly accelerated system of two particles of opposite mass. Bondi's null barrier corresponds precisely to the history of the Schwarzschild sphere of the mass m_o .

The plan of this chapter is as follows. In § 5.2 , we deal with the problem of extending the Schwarzschild manifold of a particle imbedded in a static, axially symmetric field, with particular reference to the case of n collinear particles.

Slightly generalizing a result of Mysak and Szekeres [48], we show that a trivial modification of Kruskal's transformation [54] yields an analytic extension of the manifold. The reflexional symmetry of Kruskal's transformation entails a duplication of all singularities in the extended manifold.

Finally, in § 5.3 , we carry out the limiting process just described for the extended line-element representing three collinear particles. In the limiting manifold, Kruskal's co-ordinates are asymptotically Galilean. The reflexional symmetry means that the limiting manifold represents a solution of Bondi's 4-body (rather than his 2-body) problem.

If it is true that all simply connected analytic extensions of a given analytic manifold are isometric, then our result points to the conclusion that there is no solution of Bondi's 2-body problem which is analytic everywhere outside the world-lines C_- , C_+ . A uniqueness theorem of this type has never been proved, although the isometry of the various

analytic extensions of the Schwarzschild metric obtained independently by Lemaitre [50], Robertson [51], Finkelstein [52], Fronsdal [53] and Kruskal [54] provides empirical eveidence for uniqueness, at least in the spherically symmetric case. In any events, since the history of the Schwarzschild sphere is a null 3-space - i.e., a characteristic hypersurface of the field equations - there is no compelling reason to insist on analytic extensions. Relaxation of the analyticity requirement would admit a great variety of extensions, and perhaps lead to a satisfactory global solution of Bondi's 2-body problem.

The presentation in this chapter is based in part on a paper by Israel and Khan [59] which is due to appear shortly in Nuovo Cimento. In a very recent (Feb. 1964) paper, Bonnor and Swaminarayana [24], attacking the problem from a different point of view, have displayed a global solution of Bondi's problem representing four Curzon-type particles. Their solution is included in the more general one given in § 5.3.

§ 5.2 Analytic extension of the Weyl manifold representing a Schwarzschild particle in an external field.

To pass from Weyl's static line-element φ for three collinear masses [Chapter III page 59] to a global solution of Bondi's problem, it will be necessary to extend φ within the Schwarzschild sphere of one of the positive masses.

We shall begin by considering the problem of extension for the general case of a Schwarzschild particle at rest in an external static,

axially symmetric field. If the external field is restricted to be regular everywhere on the axis of symmetry, then as shown by Mysak and Szekeres [48], a direct application of Kruskal's transformation yields an analytic extension of the manifold. Our interest is in the situation where this restriction is dropped.

We recall from Chapter II that every static, axially symmetric Einstein vacuum metric can be reduced to the canonical form

$$\Phi = e^{2(\nu-\lambda)}\,(dr^2 + dz^2) + r^2\,e^{-2\lambda}\,d\varphi^2 - e^{2\lambda}\,dt^2 \quad , \tag{5.7}$$

where the functions $\lambda(r,z)$, $\nu(r,z)$ satisfy

$$\begin{aligned} \frac{\partial\nu}{\partial r} &= r\,[(\frac{\partial\lambda}{\partial r})^2 - (\frac{\partial\lambda}{\partial z})^2] \quad , \\ \frac{\partial\nu}{\partial z} &= 2r\,\frac{\partial\lambda}{\partial r}\cdot\frac{\partial\lambda}{\partial z} \quad . \end{aligned} \tag{5.8}$$

The integrability condition of equations (5.8),

$$\nabla^2\lambda \equiv \frac{\partial^2\lambda}{\partial r^2} + \frac{1}{r}\,\frac{\partial\lambda}{\partial r} + \frac{\partial^2\lambda}{\partial z^2} = 0 \quad , \tag{5.9}$$

is formally the Newtonian potential equation.

In (5.7) let us write

$$\lambda = \lambda_o + \psi \quad , \qquad \nu = \nu_{oo} + \delta \quad , \tag{5.10}$$

where

(5.11) $$\lambda_o = \tfrac{1}{2}\log \frac{\rho_o + \rho_o' - b_o}{\rho_o + \rho_o + b_o}$$

represents the field of a Schwarzschild particle of mass $m_o > 0$, ν_{oo} (computed from λ_o by (5.8)) is given by an expression of the form (3.61) (where $m_o = \frac{1}{2}b_o$), and $\psi(r,z)$, $\delta(r,z)$ represents the external field and its interaction with the particle. Since $\nu_{oo}(0,z) = 0$ for $|z-a_o| > \frac{1}{2}b_o$, elementary flatness requires

(5.12) $$\delta(0,z) = 0$$

for portions of the z-axis not occupied by matter. On the segment $|z-a_o| < \frac{1}{2}b_o$, ψ is regular and

$$\lim_{r\to o} r\,\frac{\partial\lambda_o}{\partial z} = 0\,, \quad \lim_{r\to o} r\,\frac{\partial\lambda_o}{\partial r} = 1\,.$$

From (5.8) it then follows that δ is well-behaved on this segment and that

$$\frac{\partial(\delta-2\psi)}{\partial r} = \frac{\partial(\delta-2\psi)}{\partial z} = 0$$

(5.13) $$\text{i.e. } \delta = 2\psi = \text{constant} = -2k\,, \quad (r = 0,\ |z-a_o| < \frac{b_o}{2})$$

We pass to spherical polar co-ordinates R, θ, φ, T by the transformation

$$e^{-k}r = R(1-\frac{b}{R})^{\frac{1}{2}} \sin\theta \ ,$$

(5.14) $$e^{-k}(z-a_o) = (R-\tfrac{1}{2}b)\cos\theta \ ,$$

$$e^{k}t = T \ , \ b = b_o e^{-k} \ .$$

The metric (5.7) takes the form

(5.15) $$\Phi = e^{2(\delta-\psi+k)}\,[(1-\frac{b}{R})^{-1}\,dR^2 + R^2\,d\theta^2] + e^{-2\psi+2k}\,R^2\sin^2\theta\,d\varphi^2$$

$$- e^{2(\psi-k)}\,(1-\frac{b}{R})\,dT^2 \ ,$$

and according to (5.13), we have

(5.16) $$\delta - 2\psi + 2k = 0 \quad \text{when} \quad R = b \ .$$

Now define p, h, ζ, τ by Kruskal's formula [54] (c.f. Chapter III, page 43)

(5.17) $$1 + p = \frac{R}{b} \ , \ h^2 = 4pe^p \ ,$$

$$\zeta = bh\cosh(\frac{T}{2b}) \ , \ \tau = bh\sinh(\frac{T}{2b}) \ .$$

The transformation R, T $\rightarrow \zeta$, τ yields (for details see Appendix X)

(5.18) $$\Phi = (1+p)^{-1}\,e^{-p}\,e^{2(\psi-k)}\,(d\zeta^2-d\tau^2) + e^{2(\delta-\psi+k)}\,R^2(d\theta^2+e^{-2\delta}\sin^2\theta\,d\varphi^2)$$

$$+ \frac{e^{2(\psi-k)}}{4b^2p(1+p)e^{2p}}\,[e^{2(\delta-2\psi+2k)}-1]\,(\zeta d\zeta - \tau d\tau)^2 \ .$$

Because of (5.16) there is no singularity in this line-element at the Schwarzschild radius $R = b$, $p = 0$. We assume $\psi(R,\theta)$, $\delta(R,\theta)$ extended as analytic functions of their arguments within the Schwarzschild sphere.

The domain of validity of (5.18) is certainly contained in

$$\zeta^2 - \tau^2 > -\frac{4b^2}{e}, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi \tag{5.19}$$

owing to the singularity at $p = -1$, $R = 0$. This singularity will be the only one encountered in the interior of the Schwarzschild sphere $(R \le b)$ if the analytically extended $\psi(R,\theta)$ and $\delta(R,\theta)$ are real and regular in this region, and if the elementary flatness condition

$$\delta(R,0) = \delta(R,\pi) = 0 \qquad (0 < R \le b) \tag{5.20}$$

is satisfied.

Since (5.18) is invariant under the transformations $\zeta \to -\zeta$ and $\tau \to -\tau$, we have complete reflexional symmetry in every 2-space θ = constant., φ = constant.

The line-element (5.15) is now to ber regarded as describing the submanifold $\zeta^2 - \tau^2 > 0$ of (5.19), and is connected with (5.18) through the transformation $(\zeta,\tau) \to (R,T)$ defined by

$$R = b(1+p), \quad 4b^2 p e^p = \zeta^2 - \tau^2, \tag{5.21}$$

$$\tanh\left(\frac{T}{2b}\right) = \frac{\tau}{\zeta} \quad (\zeta^2 - \tau^2 > 0). \tag{5.22}$$

We proceed to consider the specific form assumed by (5.18) in the case where the external field is produced by two particles, with masses $m_1 < 0$, $m_2 > 0$ at rest on the z-axis. Owing to reflexional symmetry, the 2-space $\theta = 0$ exhibits four singular curves of constant R, in addition to the singularity at $R = 0$ (Fig 6).

Setting $n = 3$ in (3.52) and (3.55), we obtain from (5.10)

$$\psi = \lambda_1 + \lambda_2 \, , \quad \delta = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_{ij} + 2(\nu_{o1} + \nu_{o2}) \tag{5.23}$$

where

$$\lambda_i = \frac{m_i}{b_i} \log \frac{\rho_i + \rho_i' - b_i}{\rho_i + \rho_i' + b_i}$$

(5.24)

$$\nu_{ij} = \frac{m_i m_j}{b_i b_j} \log \frac{E(i',j)\, E(i,j')}{E(i,j)\, E(i',j')} \quad .$$

It can easily be verified that the analytically extended functions $\rho_1^{(\prime)}(R,\theta)$, $\rho_2^{(\prime)}(R,\theta)$ are real and positive throughout the region (5.19), and that

$$e^{-k}\rho_o = R - b\cos^2\frac{\theta}{2} \, , \quad e^{-k}\rho_o' = R - b\sin^2\frac{\theta}{2} \, , \tag{5.25}$$

where k is computed from (5.13) and (5.24) to be

$$k = \frac{m_1}{b_1} \log \frac{|(z_o' - z_1')|}{|(z_o' - z_1)|} + \frac{m_2}{b_2} \log \frac{|(z_o' - z_2')|}{|(z_o' - z_2)|} \tag{5.26}$$

(For details see Appendix XI).

It follows that $\psi(R,\theta)$ and $\delta(R,\theta)$ are real and regular in the region $0 \leq R \leq b$, and (5.20) is satisfied. Thus the only true singularities produced in the manifold by the Schwarzschild particle b_o are a pair of curves with equation $R = 0$. Similar remarks apply to the other two particles.

§ 5.3 Transition to Bondi's problem.

We preface this section by indicating how a "homogeneous gravitational field" (or equivalently, a uniformly accelerated reference frame) can be characterized as a limit of the Schwarzschild solution.

In Chapter III (page 39) we saw how, starting from the Minkowski line-element of flat space-time,

$$d\xi^2 + d\eta^2 + d\zeta^2 - d\tau^2 , \tag{5.27}$$

the co-ordinate transformation

$$\begin{aligned} &\xi = r_* \cos \varphi, \quad \eta = r_* \sin \varphi \\ &\zeta = z_* \sinh \left(\frac{t}{\beta}\right) , \quad \tau = z_* \cosh \left(\frac{t}{\beta}\right) \qquad (\beta = \text{constant}) \end{aligned} \tag{5.28}$$

leads to the form

$$\Phi_{ua} = dr_*^2 + r_*^2 \, d\varphi^2 + dz_*^2 - \frac{z_*^2}{\beta^2} \, dt^2 . \tag{5.29}$$

Since each point with fixed space co-ordinates r_* , z_*, φ executes hyperbolic motion, Φ_{ua} may be interpreted as the metric of flat space-time viewed from a uniformly accelerated reference frame. The motion

of this frame is not affected by a conformal transformation $r_*, z_* \to r$, z of the space co-ordinates, defined by

$$2\beta r = r_* z_* \quad , \quad 2\beta z = \tfrac{1}{2}(z_*^2 - r_*^2) \quad . \tag{5.30}$$

This has the effect of casting φ_{ua} into the Weyl canonical form (5.7), with $\lambda = \lambda_{ua}$, $\nu = \nu_{ua}$ given by

$$\lambda_{ua} = \tfrac{1}{2}\log \frac{(\rho_o + z_o)}{2\beta} \quad , \quad \nu_{ua} = \tfrac{1}{2}\log \frac{\rho_o + z_o}{2\rho_o} \tag{5.31}$$

where

$$\rho_o^2 = r^2 + z_o^2 \quad , \quad z_o = z \quad .$$

If, for comparison, we write down the Schwarzschild spherically symmetric solution $\Phi_{\text{Schw.}}$ for a single mass extending from $z = 0$ to $z = -b_o$ in the Euclidean map of Weyl's co-ordinates, we find $\lambda = \lambda_o$, $\nu = \nu_{oo}$ where (c.f. (3.21) with $m = \tfrac{1}{2}b$)

$$\lambda_o = \tfrac{1}{2}\log \frac{\rho_o + \rho_o' - b_o}{\rho_o + \rho_o' + b_o} \quad , \quad \nu_{oo} = \tfrac{1}{2}\log \frac{\rho_o\rho_o' + z_o z_o' + r^2}{\rho_o\rho_o'} \quad . \tag{5.32}$$

Now make the constant scale transformation $\Phi_{\text{Schw.}} \to \Phi'_{\text{Schw.}}$, defined by

$$g_{ij} \to g'_{ij} = \epsilon g_{ij} \ (i,j=1,2,3) \ , \ g_{44} \to g'_{44} = \epsilon^{-1} g_{44} \quad ;$$

then $\Phi'_{\text{Schw.}}$ is also a solution of the Einstein field equations. Further, set

$$b_o = \epsilon^{-1}\beta$$

and let $\epsilon \to 0$ keeping β fixed. Since

$$\lim_{\epsilon \to o} (\lambda_o - \tfrac{1}{2}\log \epsilon) = \lambda_{ua} \ , \ \lim_{\epsilon \to o} \nu_{oo} = \nu_{ua} \ ,$$

we obtain

$$(5.33) \qquad \lim_{\epsilon \to o} \Phi'_{Schw.} = \Phi_{ua} \ .$$

Yet another procedure would be to throw $\Phi_{Schw.}$ into Kruskal's form [Chapter III, page 43] before taking the limit $\epsilon \to 0$. This would lead directly to the Minkowski form (5.27). We shall use a completely analogous method to transform the 3-particle solution obtained in § 5.2.

In (5.15) and (5.18) let us set

$$-2a_o = b_o = \frac{\beta}{\epsilon} \ , \quad r = \epsilon\,\bar{r} \ , \quad z = \epsilon\,\bar{z} \ , \ a_i = \epsilon\,\bar{a}_i \ , \ b_i = \epsilon\,\bar{b}_i ,$$

$$m_i = \epsilon\,\bar{m}_i$$

where ϵ is a small parameter. Then λ_i and ν_{ij} $(i = 1,2)$ are given by the same formal expressions in the barred as in the original unbarred quantities. We shall further write

$$4\beta^2 p = \epsilon^2 p_* \ , \ 2\beta \tan\frac{\theta}{2} = \epsilon\, r_* \ ,$$

so that, according to (3.15),

$$(2\beta\bar{r})^2 = r_*^2\, p_* + O(\epsilon) ,$$

$$2\beta\bar{z} = \tfrac{1}{2}(p_* - r_*^2) + O(\epsilon) .$$

In place of θ, we shall introduce r_* as co-ordinate in the line-element (5.18).

We keep β and all barred parameters fixed and let $\epsilon \to 0$. Then $k \to 0$ by virtue of (5.26). From (5.18) we obtain a line-element which continues to satisfy the Einstein field equations:

$$(5.34) \qquad \Phi = e^{2\psi}(d\zeta^2 - d\tau^2) + e^{2(\delta-\psi)}(dr_*^2 + e^{-2\delta}\, r_*^2\, d\varphi^2)$$
$$+ \; e^{2\psi}\, [e^{2(\delta-2\psi)} - 1]\; \frac{(\zeta d\zeta - \tau d\tau)^2}{\zeta^2 - \tau^2} ,$$

where

$$(5.35) \qquad \delta = \delta\,(r_*^2 , p_*) , \; \psi = \psi(r_*^2 , p_*) ,$$

and

$$(5.36) \qquad p_* = \zeta^2 - \tau^2 ,$$

according to (5.21). Explicitly, we have (dropping all bars)

$$(5.37) \qquad \psi = \lambda_1 + \lambda_2 ,$$

$$(5.38) \qquad \delta = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_{ij} + 2(\nu_{01} + \nu_{02}) ,$$

with

$$\lambda_i = \frac{m_i}{b_i} \log \frac{\rho_i + \rho_i' - b_i}{\rho_i + \rho_i' + b_i}$$

$$\nu_{ij} = \frac{m_i m_j}{b_i b_j} \log \frac{E(i',j)\, E(i,j')}{E(i,j)\, E(i',j')}$$

and

$$\nu_{oi} = \frac{m_i}{2b_i} \log \frac{E(0,i')}{E(0,i')} \frac{\rho_i + z_i}{\rho_i' + z_i'} , \tag{5.39}$$

$$\rho_i^{(')} = [(2\beta)^{-2}\, r_*^2\, p_* + z_i^{(')2}]^{\frac{1}{2}} \qquad (i=1,2) \tag{5.40}$$

$$z_i^{(')} = (4\beta)^{-1} (p_* - r_*^2) - a_i\, (\mp)\tfrac{1}{2}b_i \quad (i=1,2) \tag{5.41}$$

$$2\beta z_o = \tfrac{1}{2}(p_* - r_*^2)\ ,\ 2\beta\rho_o = \tfrac{1}{2}(p_* + r_*^2) \quad . \tag{5.42}$$

Since δ and ψ tend to zero when $\zeta^2 + r_*^2 \to \infty$ for fixed τ, the co-ordinates r_*, φ, ζ are asymptotically cylindrical. We recall from (5.13) that

$$\delta - 2\psi = 0\ (p_*) \quad (p_* \to 0) ,$$

so that there is no singularity on the null cone $\zeta^2 - \tau^2 = 0$. In fact, the only singularities involved in (5.34) are the histories of the two "rods" on the z-axis:

$$r_* = 0\ ,\ |(4\beta)^{-1} (\zeta^2 - \tau^2) - a_i\ | \le \tfrac{1}{2}|b_i| \quad (i = 1,2,). \tag{5.43}$$

One checks immediately that the elementary flatness condition ($\delta = 0$ when $r_* = 0$) is satisfied everywhere outside these singular regions, provided

$$(5.44)\qquad \frac{m_1}{b_1}\log\frac{a_1-\frac{1}{2}b_1}{a_1+\frac{1}{2}b_1} = \frac{m_2}{b_2}\log\frac{a_2+\frac{1}{2}b_2}{a_2-\frac{1}{2}b_2} = \frac{2m_1m_2}{b_1b_2}\log\left|1-\frac{b_1\,b_2}{(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2}\right| .$$

This condition follows if we write the elementary-flatness condition (3.70), for three particles, numbered 0, 1, 2 and let $m_o = \frac{1}{2}b_o = -a_o \to \infty$ (for details see Appendix XII).

The arguments of §5.2 show that each of the singularities (5.43) is removable by a co-ordinate transformation. The only true singularities are the four time-like curves

$$r_* = 0,\quad \zeta^2 - \tau^2 = 4\beta a_i \qquad (i = 1,2) \ .$$

It is now clear that <u>(5.34) represents an explicit analytic solution of the Bondi 4-body problem described in §5.2</u>. We observe from (5.44) that the masses of the two components of Bondi dipole are not precisely equal and opposite, but proportional to the inverse square root of the respective accelerations (as measured by the curvature of the world-lines).

APPENDIX I

CALCULATION OF RICCI TENSOR CORRESPONDING TO THE METRIC FORM (2.33).

Following are the detailed calculations of the components of the Ricci tensor corresponding to the line-element (2.33) (in Chapter II page 22).

We have

(1) $$ds^2 = A^2[(dx^1)^2 + (dx^2)^2] + B^2(dx^3)^2 - C^2(dx^4)^2$$

where A, B, and C are functions of (x^1, x^2) only.

(2) $$g_{11} = g_{22} = A^2 \quad ; \quad g_{33} = B^2 \text{ and } g_{44} = -C^2$$
$$g^{11} = g^{22} = A^{-2} \quad ; \quad g^{33} = B^{-2} \text{ and } g^{44} = -C^{-2}$$

Ricci tensor is defined by

(3) $$R_{ij} = R^k_{ijk} = g^{k\mu} R_{kij\mu} = R_{ji} \quad .$$

Explicitly we have

(4) $$R_{ij} = \Gamma^a_{ai,j} - \Gamma^a_{ij,a} + \Gamma^a_{bi} \Gamma^b_{aj} - \Gamma^a_{ij} \Gamma^b_{ab}$$

$$= \tfrac{1}{2}[\log(-g)]_{,ij} - \Gamma^a_{ij,a} - \tfrac{1}{2}\Gamma^a_{ij}[\log(-g)]_{,a} + \Gamma^a_{bi} \Gamma^b_{aj}$$

The non-vanishing components of Ricci tensor corresponding to (1) are as follows.

$$R_{11} = \partial_1(\frac{1}{A}\partial_1 A) + \partial_2(\frac{1}{A}\partial_2 A) + (\frac{1}{B}\partial^2_{11}B + \frac{1}{C}\partial^2_{11}C)$$

$$+ \frac{1}{A}\partial_2 A(\frac{1}{B}\partial_2 B + \frac{1}{C}\partial_2 C) - \frac{1}{A}\partial_1 A(\frac{1}{B}\partial_1 B + \frac{1}{C}\partial_1 C)$$

$$R_{22} = \partial_1(\frac{1}{A}\partial_1 A) + \partial_2(\frac{1}{A}\partial_2 A) + (\frac{1}{B}\partial^2_{22}B + \frac{1}{C}\partial^2_{22}C)$$

$$+ \frac{1}{A}\partial_1 A(\frac{1}{B}\partial_1 B + \frac{1}{C}\partial_1 C) - \frac{1}{A}\partial_2 A(\frac{1}{B}\partial_2 B + \frac{1}{C}\partial_2 C) .$$

(5) $$R_{12} = \frac{1}{B}\partial^2_{12}B + \frac{1}{C}\partial^2_{12}C - \frac{1}{A}\partial_2 A(\frac{1}{B}\partial_1 B + \frac{1}{C}\partial_1 C)$$

$$- \frac{1}{A}\partial_1 A(\frac{1}{B}\partial_2 B + \frac{1}{C}\partial_2 C)$$

$$R_{33} = \frac{B}{A^2}[\partial^2_{11}B + \partial^2_{22}B + \frac{1}{C}(\partial_1 B\partial_1 C + \partial_2 B\partial_2 C)]$$

$$R_{44} = \frac{-C}{A^2}[\partial^2_{11}C + \partial^2_{22}C + \frac{1}{B}(\partial_1 B\,\partial_1 C + \partial_2 B\,\partial_2 C)]$$

Since

$$g^{ij}R_{ij} = R^i_i$$

(6) $$R^3_3 = g^{33}R_{33} = B^{-2}R_{33}$$

$$(7) \qquad R^4_4 = g^{44} R_{44} = -C^{-2} R_{44} \quad .$$

Hence

$$(8) \qquad R^3_3 + R^4_4 = \frac{R_{33}}{B^2} - \frac{R_{44}}{C^2} = \frac{1}{A^2BC} [\partial^2_{11} (BC) + \partial^2_{22} (BC)] \quad .$$

In a matter free domain, the field equations are

$$R_{ij} = 0 \quad .$$

Hence (8) becomes

$$R^3_3 + R^4_4 = \frac{1}{A^2BC} [\partial^2_{11} (BC) + \partial^2_{22} (BC)] = 0$$

i.e. (BC) is a harmonic function of (x^1, x^2) .

$$\partial_1 = \frac{\partial}{\partial x^1} \quad , \quad \partial_2 = \frac{\partial}{\partial x^2} \quad , \quad \partial^2_{11} = \frac{\partial^2}{\partial x_1^2} \quad , \quad \partial^2_{22} = \frac{\partial^2}{\partial x_2^2} \quad .$$

APPENDIX II

CO-ORDINATE TRANSFORMATION OF THE METRIC (3.1).

Following are the detailed calculations for the co-ordinate transformation of the line-element (3.1) (c.f. Chapter III, p.43).

From (3.1), we have

(1) $$\varphi = e^{2(\nu-\lambda)}(dr^2+dz^2)+r^2 e^{-2\lambda} d\varphi^2 - e^{2\lambda}\, dt^2$$

where

(2) $$\lambda = \frac{m}{b} \log \frac{\rho + \rho' - b}{\rho + \rho' + b}$$

(3) $$\nu = \frac{2m^2}{b^2} \log \frac{(\rho + \rho')^2 - b^2}{4\rho\rho'}$$

(4) $$\rho^2 = r^2 + z^2 \quad , \quad \rho^{12} = r^2 + (z-b)^2 \quad .$$

Set

(5) $$r = R(1 - \frac{b}{R})^{\frac{1}{2}} \sin \theta$$

(6) $$z - \tfrac{1}{2}b = (R - \tfrac{1}{2}b) \cos \theta$$

(7) $$m = \tfrac{1}{2}b \quad .$$

Hence

$$(8) \qquad dr = \frac{\sin\theta(2R-b)}{2R(1-\frac{b}{R})^{\frac{1}{2}}} dR + R(1-\frac{b}{R})^{\frac{1}{2}} \cos\theta \, d\theta$$

$$(9) \qquad dz = \cos\theta \, dR - (R - \tfrac{1}{2}b) \sin\theta \, d\theta \quad .$$

Squaring and adding (8) and (9) we have

$$(10) \qquad (dr^2 + dz^2) = \frac{R^2 - bR + \frac{b^2}{4} \sin\theta}{R^2} \left[\frac{dR^2}{(1-\frac{b}{R})} + R^2 d\theta^2 \right] \quad .$$

From (2) and (7) we get

$$(11) \qquad \lambda = \tfrac{1}{2}\log \frac{\rho + \rho' - b}{\rho + \rho' + b}$$

$$(12) \qquad \nu = \tfrac{1}{2}\log \frac{(\rho + \rho')^2 - b^2}{4\rho\rho'} \quad .$$

From (5) and (6)

$$(13) \qquad \begin{aligned} \rho^2 &= (R^2 - bR + \frac{b^2}{4}) + \frac{b^2}{4}\cos^2\theta + b\cos\theta(R - \tfrac{1}{2}b) \\ &= (R - \tfrac{1}{2}b)^2 + \frac{b^2}{4}\cos^2\theta + b\cos\theta(R - \tfrac{1}{2}b) \end{aligned}$$

$$(14) \qquad \rho'^2 = (R - \tfrac{1}{2}b)^2 + \frac{b^2}{4}\cos^2\theta - b\cos\theta(R - \tfrac{1}{2}b)$$

$$\rho^2\rho'^2 = \{(R - \tfrac{1}{2}b)^2 + \frac{b^2}{4}\cos^2\theta\}^2 - b^2\cos^2\theta(R - \tfrac{1}{2}b)^2$$

$$\rho^2\rho'^2 = \{(R - \tfrac{1}{2}b)^2 - \frac{b^2}{4}\cos^2\theta\}^2 \tag{15}$$

$$\rho\rho' = \{(R - \tfrac{1}{2}b)^2 - \frac{b^2}{4}\cos^2\theta\} \quad . \tag{16}$$

From (13), (14) and (16), we have

$$\left.\begin{aligned}(\rho+\rho')^2 &= 4(R - \frac{b}{2})^2 \\ (\rho+\rho') &= 2(R - \frac{b}{2}) \quad .\end{aligned}\right. \tag{17}$$

Hence (11) and (12) can be written as

$$e^{2\lambda} = \frac{\rho + \rho' - b}{\rho + \rho' + b} = (1 - \frac{b}{R}) \tag{18}$$

$$e^{2\nu} = \frac{(\rho + \rho')^2 - b^2}{4\rho\rho'} = \frac{R^2(1 - \frac{b}{R})}{(R^2 - bR + \frac{b^2}{4}\sin^2\theta)} \quad . \tag{19}$$

From (18) and (19)

$$e^{2(\nu-\lambda)} = \frac{R^2}{(R^2 - bR + \frac{b^2}{4}\sin^2\theta)} \quad . \tag{20}$$

Hence from (10), (18) and (20)

(21) $$e^{2(\nu-\lambda)}(dr^2 + dz^2) = \left(\frac{dR^2}{(1-\frac{b}{R})} + R^2 d\theta^2 \right)$$

and

(22) $$r^2 e^{-2\lambda} = R^2 \sin^2 \theta \quad .$$

Hence (1) can be written as

$$e^{2(\nu-\lambda)}(dr^2 + dz^2) + r^2 e^{-2\lambda}\, d\varphi^2 - e^{2\lambda}\, dt^2$$

$$= \frac{dR^2}{(1-\frac{b}{R})} + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - (1 - \frac{b}{R})\, dt^2 \quad .$$

APPENDIX III

VERIFICATION OF EQUATION (3.37).

Following are the detailed calculations for the verification of equations (3.37a) and (3.37b) (c.f. Chapter III, page 46).

Using the Levi-Civita equations

(1) $$\frac{\partial \nu}{\partial r} = r\left[\left(\frac{\partial \lambda}{\partial r}\right)^2 - \left(\frac{\partial \lambda}{\partial z}\right)^2\right]$$

(2) $$\frac{\partial \nu}{\partial z} = 2r\,\frac{\partial \lambda}{\partial r}\,\frac{\partial \lambda}{\partial z}$$

we write

(3)
$$\frac{\partial \nu}{\partial \xi} = r\left[\left(\frac{\partial \lambda}{\partial r}\right)^2 - \left(\frac{\partial \lambda}{\partial z}\right)^2\right]\frac{\partial r}{\partial \xi} + 2r\,\frac{\partial \lambda}{\partial r}\,\frac{\partial \lambda}{\partial z}\,\frac{\partial z}{\partial \xi}$$

$$\frac{\partial \nu}{\partial \eta} = r\left[\left(\frac{\partial \lambda}{\partial r}\right)^2 - \left(\frac{\partial \lambda}{\partial z}\right)^2\right]\frac{\partial r}{\partial \eta} + 2r\,\frac{\partial \lambda}{\partial r}\,\frac{\partial \lambda}{\partial z}\,\frac{\partial z}{\partial \eta}$$

or

(4) $$\frac{\partial \nu}{\partial \xi} = r\left[\left(\lambda_\xi \frac{\partial \xi}{\partial r} + \lambda_\eta \frac{\partial \eta}{\partial r}\right)^2 - \left(\lambda_\xi \frac{\partial \xi}{\partial z} + \lambda_\eta \frac{\partial \eta}{\partial z}\right)^2\right]\frac{\partial r}{\partial \xi}$$

$$+ 2r\left[\left(\lambda_\xi \frac{\partial \xi}{\partial r} + \lambda_\eta \frac{\partial \eta}{\partial r}\right)\left(\lambda_\xi \frac{\partial \xi}{\partial z} + \lambda_\eta \frac{\partial \eta}{\partial z}\right)\right]\frac{\partial z}{\partial \xi}$$

and

(5) $$\frac{\partial \nu}{\partial \eta} = r\left[\left(\lambda_\xi \frac{\partial \xi}{\partial r} + \lambda_\eta \frac{\partial \eta}{\partial r}\right)^2 - \left(\lambda_\xi \frac{\partial \xi}{\partial z} + \lambda_\eta \frac{\partial \eta}{\partial z}\right)^2\right] \frac{\partial r}{\partial \eta}$$

$$+ 2r\left[\left(\lambda_\xi \frac{\partial \xi}{\partial r} + \lambda_\eta \frac{\partial \eta}{\partial r}\right)\left(\lambda_\xi \frac{\partial \xi}{\partial z} + \lambda_\eta \frac{\partial \eta}{\partial z}\right)\right] \frac{\partial z}{\partial \eta}$$

where $\lambda_\xi = \frac{\partial \lambda}{\partial \xi}$, $\lambda_\eta = \frac{\partial \lambda}{\partial \eta}$.

From

(6) $$r = b(\xi^2 + 1)^{\frac{1}{2}}(1 - \eta^2)^{\frac{1}{2}} , \; z = b\,\xi\,\eta ,$$

we have

(7) $$dr = \frac{b\xi(1 - \eta^2)^{\frac{1}{2}}}{(\xi^2+1)^{\frac{1}{2}}}\, d\xi - \frac{b\eta(\xi^2 + 1)^{\frac{1}{2}}}{(1-\eta^2)^{\frac{1}{2}}}\, d\eta$$

and

(8) $$dz = b\eta\, d\xi + b\xi\, d\eta .$$

Hence from (7) and (8)

(9) $$\frac{\partial r}{\partial \xi} = b\xi \frac{(1 - \eta^2)^{\frac{1}{2}}}{(\xi^2+1)^{\frac{1}{2}}} , \quad \frac{\partial r}{\partial \eta} = - b\eta \frac{(\xi^2 + 1)^{\frac{1}{2}}}{(1 - \eta^2)^{\frac{1}{2}}}$$

(10) $$\frac{\partial z}{\partial \xi} = b\eta , \quad \frac{\partial z}{\partial \eta} = b\xi .$$

Setting $dz = 0$ in (8), from (7) and (8) we have

$$dr = b\xi \frac{(1 - \eta^2)^{\frac{1}{2}}}{(\xi^2 + 1)^{\frac{1}{2}}} d\xi - b\eta \frac{(\xi^2 + 1)^{\frac{1}{2}}}{(1 - \eta^2)^{\frac{1}{2}}} d\eta$$

(11)

$$0 = b\eta \, d\xi + b\xi \, d\eta \quad .$$

Solving (11) for $d\xi$ and $d\eta$ we have

$$\left(\frac{\partial \xi}{\partial r}\right)_{dz=o} = \frac{\xi(\xi^2 + 1)^{\frac{1}{2}}(1 - \eta^2)^{\frac{1}{2}}}{b(\xi^2 + \eta^2)}$$

(12)

$$\left(\frac{\partial \eta}{\partial r}\right)_{dz=o} = \frac{-\eta(\xi^2 + 1)^{\frac{1}{2}}(1 - \eta^2)^{\frac{1}{2}}}{b(\xi^2 + \eta^2)} \quad .$$

Next by setting $dr = 0$ in (8) we have

$$\left(\frac{\partial \xi}{\partial z}\right)_{dr=o} = \frac{\eta(\xi^2 + 1)}{b(\xi^2 + \eta^2)}$$

(13)

$$\left(\frac{\partial \eta}{\partial z}\right)_{dr=o} = \frac{\xi(1 - \eta^2)}{b(\xi^2 + \eta^2)} \quad .$$

From (4) we get

$$\frac{\partial \nu}{\partial \xi} = r\lambda_\xi^2\left[\left\{\left(\frac{\partial \xi}{\partial r}\right)^2 - \left(\frac{\partial \xi}{\partial z}\right)^2\right\}\frac{\partial r}{\partial \xi} + 2\,\frac{\partial \xi}{\partial r}\frac{\partial \xi}{\partial z}\frac{\partial z}{\partial \xi}\right] +$$

$$+ r\lambda_\eta^2\left[\left\{\left(\frac{\partial \eta}{\partial r}\right)^2 - \left(\frac{\partial \eta}{\partial z}\right)^2\right\}\frac{\partial r}{\partial \xi} + 2\,\frac{\partial \eta}{\partial r}\,\frac{\partial \eta}{\partial z}\,\frac{\partial z}{\partial \xi}\right] -$$

$$-2r\lambda_\xi\lambda_\eta\left[\left(\frac{\partial \xi}{\partial z}\frac{\partial \eta}{\partial z} - \frac{\partial \xi}{\partial r}\frac{\partial \eta}{\partial r}\right)\frac{\partial r}{\partial \xi} - \left(\frac{\partial \xi}{\partial r}\frac{\partial \eta}{\partial z} + \frac{\partial \eta}{\partial r}\frac{\partial \xi}{\partial z}\right)\frac{\partial z}{\partial \xi}\right] \quad .$$

From (9), (10), (12) and (13), we get

$$(14)\qquad r\left\{\left(\frac{\partial\xi}{\partial r}\right)^2 - \left(\frac{\partial\xi}{\partial z}\right)^2\right\}\frac{\partial r}{\partial\xi} = \frac{\xi(\xi^2+1)(1-\eta^2)(\xi^2-2\eta^2\xi^2-\eta^2)}{(\xi^2+\eta^2)^2}$$

$$(15)\qquad 2r\,\frac{\partial\xi}{\partial r}\,\frac{\partial\xi}{\partial z}\,\frac{\partial z}{\partial\xi} = \frac{2\xi\eta^2(\xi^2+1)^2(1-\eta^2)}{(\xi^2+\eta^2)^2}$$

Hence

$$(16)\qquad r\left[\left\{\left(\frac{\partial\xi}{\partial r}\right)^2 - \left(\frac{\partial\xi}{\partial z}\right)^2\right\}\frac{\partial r}{\partial\xi} + 2\,\frac{\partial\xi}{\partial r}\,\frac{\partial\xi}{\partial z}\,\frac{\partial z}{\partial\xi}\right] = \frac{\xi(\xi^2+1)(1-\eta^2)}{(\xi^2+\eta^2)} .$$

Also

$$(17)\qquad r\left\{\left(\frac{\partial\eta}{\partial r}\right)^2 - \left(\frac{\partial\eta}{\partial z}\right)^2\right\}\frac{\partial r}{\partial\xi} = \frac{\xi(1-\eta^2)^2}{(\xi^2+\eta^2)^2}\,(\eta^2+2\xi^2\eta^2-\xi^2)$$

$$(18)\qquad 2r\,\frac{\partial\eta}{\partial r}\,\frac{\partial\eta}{\partial z}\,\frac{\partial z}{\partial\xi} = -\frac{2\eta^2\xi(\xi^2+1)(1-\eta^2)^2}{(\xi^2+\eta^2)} .$$

Hence from (17) and (18)

$$(19)\qquad r\left[\left\{\left(\frac{\partial\eta}{\partial r}\right)^2 - \left(\frac{\partial\eta}{\partial z}\right)^2\right\}\frac{\partial r}{\partial\xi} + 2\,\frac{\partial\eta}{\partial r}\,\frac{\partial\eta}{\partial z}\,\frac{\partial z}{\partial\xi}\right] = -\frac{\xi(1-\eta^2)^2}{(\xi^2+\eta^2)} .$$

Again from (9), (10), (12) and (13)

$$(20)\qquad r\left(\frac{\partial\xi}{\partial z}\,\frac{\partial\eta}{\partial z} - \frac{\partial\xi}{\partial r}\,\frac{\partial\eta}{\partial r}\right)\frac{\partial r}{\partial\xi} = \frac{2\xi^2\eta(\xi^2+1)(1-\eta^2)^2}{(\xi^2+\eta^2)^2}$$

and

$$(21) \qquad r\left(\frac{\partial\xi}{\partial r}\,\frac{\partial\eta}{\partial z} + \frac{\partial\eta}{\partial r}\,\frac{\partial\xi}{\partial z}\right)\frac{\partial z}{\partial\xi} = \eta(\xi^2 + 1)(1 - \eta^2)(\xi^2 - 2\xi^2\eta^2 - \eta^2)$$

Hence from (20) and (21)

$$(22) \qquad -2r\left[\left(\frac{\partial\xi}{\partial z}\,\frac{\partial\eta}{\partial z} - \frac{\partial\xi}{\partial r}\,\frac{\partial\eta}{\partial r}\right)\frac{\partial r}{\partial\xi} - \left(\frac{\partial\xi}{\partial r}\,\frac{\partial\eta}{\partial z} + \frac{\partial\eta}{\partial r}\,\frac{\partial\xi}{\partial z}\right)\frac{\partial z}{\partial\xi}\right]$$

$$= \frac{-2\eta(\xi^2 + 1)(1 - \eta^2)}{(\xi^2 + \eta^2)} \, .$$

Finally from (4), (16), (19) and (22)

$$(23) \qquad \frac{\partial\nu}{\partial\xi} = \frac{\xi(\xi^2+1)(1-\eta^2)}{(\xi^2+\eta^2)}\left(\frac{\partial\lambda}{\partial\xi}\right)^2 - \frac{\xi(1-\eta^2)^2}{(\xi^2+\eta^2)}\left(\frac{\partial\lambda}{\partial\eta}\right)^2 - \frac{2\eta(\xi^2+1)(1-\eta^2)}{(\xi^2+\eta^2)}\,\frac{\partial\lambda}{\partial\xi}\,\frac{\partial\lambda}{\partial\eta} \, .$$

In exactly the same way it can be shown that

$$(24) \qquad \frac{\partial\nu}{\partial\eta} = \frac{\eta(\xi^2+1)^2}{(\xi^2+\eta^2)}\left(\frac{\partial\lambda}{\partial\xi}\right)^2 - \frac{\eta(1-\eta^2)(\xi^2+1)}{(\xi^2 + \eta^2)}\left(\frac{\partial\lambda}{\partial\eta}\right)^2 - \frac{2\xi(\xi^2+1)(1-\eta^2)}{(\xi^2+\eta^2)}\,\frac{\partial\lambda}{\partial\xi}\,\frac{\partial\lambda}{\partial\eta} \, .$$

APPENDIX IV

FIELDS OF OBLATE AND PROLATE HOMOGENEOUS SOLID SPHEROIDS.

With reference to Chapter III page 50, following are the detailed calculations for the field of oblate and prolate homogeneous solid spheroids.

I. Oblate spheroidal co-ordinates are defined by

(1) $$r = b(1+\xi^2)^{\frac{1}{2}}\ (1-\eta^2)^{\frac{1}{2}}$$

(2) $$z = b\ \xi\ \eta\ .$$

Hence

(3) $$dr = b\{\frac{\xi(1-\eta^2)d\xi\ -\ \eta(\xi^2+1)\ d\eta}{(\xi^2+1)^{\frac{1}{2}}(1-\eta^2)^{\frac{1}{2}}}\}$$

(4) $$dz = b\{\eta\ d\ \xi\ +\ \xi\ d\ \eta\}\ .$$

If λ represents the Newtonian potential of an oblate homogeneous spheroid then

(5) $$\lambda = A[4\ \cot^{-1}\xi\ +\ (3\eta^2-1)\ \{(1+3\xi^2)\ \cot^{-1}\xi\ -\ 3\xi]$$

where A is a constant connected with the mass of the spheroid

(6) $$\frac{\partial \nu}{\partial r} = r[(\frac{\partial\lambda}{\partial r})^2 - (\frac{\partial\lambda}{\partial z})^2]$$

(7) $$\frac{\partial \nu}{\partial z} = 2r \frac{\partial \lambda}{\partial r} \frac{\partial \lambda}{\partial z} \quad .$$

From (6) and (7)

(8) $$\begin{aligned} d\nu &= \frac{\partial \nu}{\partial r} dr + \frac{\partial \nu}{\partial z} dz \\ &= r[(\frac{\partial \lambda}{\partial r})^2 - (\frac{\partial \lambda}{\partial z})^2] dr + 2r\frac{\partial \lambda}{\partial r} \frac{\partial \lambda}{\partial z} dz \quad . \end{aligned}$$

But

$$r\{ (\frac{\partial \lambda}{\partial r})^2 - (\frac{\partial \lambda}{\partial z})^2\} dr \quad \text{can be written as}$$

(9) $$\begin{aligned} & r\left[(\frac{\partial \lambda}{\partial r})^2 - (\frac{\partial \lambda}{\partial z})^2\right] dr \\ &= A^2\Big[\{\xi^3(1-\eta^2)(1-5\eta^2) + \xi(1-\eta^2)^2\} (\cot^{-1}\xi)^2 \\ &\quad - 2\{\xi^2(1-5\eta^2)(1-\eta^2)\} (\cot^{-1}\xi) + \frac{\xi^3(1-5\eta^2)(1-\eta^2)-4\xi\eta^2(1-\eta^2)}{(\xi^2+1)}\Big] d\xi \\ &\quad - A^2\Big[\{\xi^2(\xi^2+1)\eta(1-5\eta^2) + \eta(1-\eta^2)(\xi^2+1)\} (\cot^{-1}\xi)^2 \\ &\quad - 2\{\xi\eta(\xi^2+1)(1-5\eta^2)\} \cot^{-1}\xi + \{\xi^2\eta(1-5\eta^2)-4\eta^3\}\Big] d\eta \end{aligned}$$

and

$$(10) \qquad 2r \frac{\partial\lambda}{\partial r} \frac{\partial\lambda}{\partial z}\, dz$$

$$= - A^2\left[4\xi\eta^2(\xi^2+1)(1-\eta^2)(\cot^{-1}\xi)^2 - 4\eta^2(1-\eta^2)(2\xi^2+1)\cot^{-1}\xi + 4\xi\eta^2(1-\eta^2)\right]d\xi$$

$$- A^2\left[4\xi^2\eta(\xi^2+1)(1-\eta^2)(\cot^{-1}\xi)^2 - 4\xi\eta(1-\eta^2)(2\xi^2+1)\cot^{-1}\xi + 4\xi^2\eta(1-\eta^2)\right]d\eta$$

(8) becomes

$$(11) \qquad d\nu = A^2\Bigg[\{\xi^3 - 10\xi^3\eta^2 + 9\xi^3\eta^4 - 6\xi\eta^2 + 5\xi\eta^4 + \xi\}(\cot^{-1}\xi)^2 +$$

$$+ \{20\xi^2\eta^2 - 18\xi^2\eta^4 - 2\xi^2 + 4\eta^2 + 4\eta^4\}(\cot^{-1}\xi)$$

$$+ \frac{\{\xi^3 - 10\xi^3\eta^2 + 9\xi^3\eta^4 - 8\xi\eta^2 + 8\xi\eta^4\}}{(\xi^2+1)}\Bigg]\, d\xi \quad +$$

$$+ \Bigg[\{9\xi^4\eta^3 - 5\xi^4\eta - 6\xi^3\eta + 10\xi^2\eta^3 + \eta^3 - \eta\}(\cot^{-1}\xi)^2$$

$$+ \{10\xi^3\eta - 18\xi^3\eta^3 + 6\xi\eta - 14\xi\eta^3\}(\cot^{-1}\xi)$$

$$+ \{9\xi^2\eta^3 - 5\xi^2\eta - 4\eta^3\}\Bigg]\, d\eta \qquad .$$

Hence

$$(12) \qquad \frac{\partial\nu}{\partial\xi} = A^2 \left[\xi^3 - 10\xi^3\eta^2 + 9\xi^3\eta^4 - 6\xi\eta^2 + 5\xi\eta^4 + \xi\right](\cot^{-1}\xi)^2 +$$

$$+ \left[20\xi^2\eta^2 - 18\xi^2\eta^4 - 2\xi^2 + 4\eta^2 - 4\eta^4\right](\cot^{-1}\xi) \quad +$$

$$+ \frac{[\xi^3 - 10\xi^3\eta^2 + 9\xi^3\eta^4 - 8\xi\eta^2 + 8\xi\eta^4]}{(\xi^2+1)}$$

and

(13) $$\frac{\partial \nu}{\partial \eta} = A^2\Big\{ \Big[9\xi^4\eta^3 - 5\xi^4\eta - 6\xi^2\eta + 10\xi^2\eta^3 + \eta^3 - \eta\Big](\cot^{-1}\xi)^2$$

$$+ \Big[10\xi^3\eta - 18\xi^3\eta^3 + 6\xi\eta - 14\xi\eta^3\Big](\cot^{-1}\xi) + \Big[9\xi^2\eta^3 - 5\xi^2\eta + 4\eta^3\Big]\Big\} .$$

For the determination of ν from (12), let us consider the evaluation of the following integrals

(i) $$\int \xi^3(\cot^{-1}\xi)^2 d\xi$$

$$= \frac{\xi^4}{4}(\cot^{-1}\xi)^2 + \frac{1}{2}\int \frac{\xi^4\cot^{-1}\xi}{\xi^2+1}\, d\xi$$

$$= \frac{\xi^4}{4}(\cot^{-1}\xi)^2 + \frac{1}{2}\int \frac{\xi^4 + \xi^2 - \xi^2 - 1 + 1}{(\xi^2+1)}\cot^{-1}\xi\, d\xi$$

$$= \frac{\xi^4}{4}(\cot^{-1}\xi)^2 + \frac{1}{2}\int \xi^2\cot^{-1}\xi\, d\xi - \frac{1}{2}\int \cot^{-1}\xi\, d\xi + \frac{1}{2}\int \frac{\cot^{-1}\xi}{(\xi^2+1)}\, d\xi$$

$$= \frac{\xi^4}{4}(\cot^{-1}\xi)^2 + \frac{1}{2}\Big\{\frac{\xi^3}{3}\cot^{-1}\xi + \frac{1}{3}\int\frac{\xi^3}{(\xi^2+1)}\,d\xi\Big\} - \frac{1}{2}\Big[\xi\cot^{-1}\xi + \int\frac{\xi\, d\xi}{\xi^2+1}\Big]$$
$$- \frac{1}{4}(\cot^{-1}\xi)^2$$

$$= \frac{\xi^4}{4}(\cot^{-1}\xi)^2 + \frac{1}{6}\xi^3\cot^{-1}\xi + \frac{1}{6}\int\Big\{\xi - \frac{\xi}{(\xi^2+1)}\Big\}\,d\xi - \frac{1}{2}\xi\cot^{-1}\xi - \frac{1}{4}\log(\xi^2+1)$$
$$- \frac{1}{4}(\cot^{-1}\xi)^2$$

$$= \frac{1}{4}(\xi^4-1)(\cot^{-1}\xi)^2 + \frac{1}{6}\xi(\xi^2-3)\cot^{-1}\xi - \frac{1}{3}\log(\xi^2+1) + \frac{\xi^2}{12} .$$

(ii) In exactly the same way we get

$$\int \xi(\cot^{-1}\xi)^2 d\xi = (\frac{\xi^2+1}{2})(\cot^{-1}\xi)^2 + \xi \cot^{-1}\xi + \frac{1}{2}\log(\xi^2+1)$$

$$\int \xi^2 \cot^{-1}\xi \, d\xi = \frac{\xi^3}{3}\cot^{-1}\xi + \frac{1}{6}\xi^2 - \frac{1}{6}\log(\xi^3+1)$$

and

$$\int \frac{\xi^3}{(\xi^2+1)} = \frac{\xi^2}{2} - \frac{1}{2}\log(\xi^2+1) \quad .$$

Hence

$$(14) \qquad \nu = A^2 \quad \frac{1}{4}(\xi^2+1)\left[\xi^2(1-10\eta^2+9\eta^4) + (1-\eta^2)^2\right](\cot^{-1}\xi)^2$$

$$+ \frac{\xi}{2}\left[(1-6\eta^2-7\eta^4) - \xi^2(1-10\eta^2+9\eta^4)\right]\cot^{-1}\xi$$

$$+ \frac{\xi^2}{4}\left[1 - 10\eta^2 + 9\eta^4\right] \quad + f(\eta)$$

where $f(\eta)$ is an arbitrary function of η .

Differentiating (14) partially with respect to η

$$(15) \qquad \frac{\partial \nu}{\partial \eta} = A^2\left\{\left[9\xi^4\eta^3-5\xi^4\eta+10\xi^2\eta^3-6\xi^2\eta+\eta^3-\eta\right](\cot^{-1}\xi)^2 + \left[-18\xi^3\eta^3+10\xi^3\eta-14\xi\eta^3+6\xi\eta\right](\cot^{-1}\xi)+\left[9\xi^2\eta^3-5\xi^2\eta\right]\right\}+f'(\eta)$$

From (15) and (13) we have

$$(16) \qquad f'(\eta) = 4A^2\eta^3 \quad .$$

Integrating we get

$$(17) \qquad f(\eta) = A^2\eta^4 + A^2 C$$

where C is a constant of integration.

From (14) and (17)

$$(18) \qquad \nu = A^2\left\{\frac{1}{4}(\xi^2+1)\left[\xi^2(1-10\eta^2+9\eta^4)+(1-\eta^2)^2\right](\cot^{-1}\xi)^2 + \frac{\xi}{2}\left[(1+6\eta^2-7\eta^4)-\xi^2(1-10\eta^2+9\eta^4)\right](\cot^{-1}\xi) + \frac{\xi^2}{4}\left[1-10\eta^2+9\eta^4\right]+\eta^4+C\right\} .$$

The constant of integration C can be calculated from the condition that the line element becomes flat at infinity i.e. $\nu \to 0$ when $r = 0$, $z \to \infty$ corresponding to $\xi \to \infty$, $\eta = 1$. This condition is satisfied only if $C = -1$.

Hence,

$$(19) \qquad \nu = A^2\left\{\frac{1}{4}(\xi^2+1)\left[\xi^2(1-10\eta^2+9\eta^4)+(1-\eta^2)^2\right](\cot^{-1}\xi)^2 + \right.$$

$$\left. + \frac{\xi}{2}\left[(1+6\eta^2-7\eta^4)-\xi^2(1-10\eta^2+9\eta^4)\right](\cot^{-1}\xi) + \frac{\xi^2}{4}\left[1-10\eta^2+9\eta^4\right]+\eta^4-1\right\}$$

or

$$\nu = 9\alpha^2\eta^4\left[(1+\xi^2)(1+9\xi^2)(\cot^{-1}\xi)^2-(14\xi+18\xi^3)(\cot^{-1}\xi)+9\xi^2+4\right]$$

$$- 18\alpha^2\eta^2\left[(1+\xi^2)(1+5\xi^2)(\cot^{-1}\xi)^2-(6\xi+10\xi^3)(\cot^{-1}\xi)+5\xi^2\right]$$

$$+ 9\alpha^2\left[(1+\xi^2)^2(\cot^{-1}\xi)^2+2(\xi-\xi^3)\cot^{-1}\xi+\xi^2-4\right]$$

(where $\alpha^2 = 4A^2$) .

This result agrees with Misra's [27] result of oblate homogeneous spheroid.

II. Prolate spheroidal co-ordinates are defined by (c.f. Chapter III, (3.42))

$$(20) \qquad r = b(1-\xi^2)^{\frac{1}{2}}(\eta^2-1)^{\frac{1}{2}}$$

$$(21) \qquad z = b\,\xi\,\eta \quad .$$

Hence

(22) $$dr = \frac{b\{\eta(1-\xi^2)d\eta - \xi(\eta^2-1)\ d\xi\}}{(1-\xi^2)^{\frac{1}{2}}(\eta^2-1)^{\frac{1}{2}}}$$

(23) $$dz = b\{\eta d\xi + \xi d\eta\} \quad .$$

If λ represents the Newtonian potential of a prolate homogeneous spheroid then

(24) $$\lambda = K[4\coth^{-1}\eta-(3\xi^2-1)\{(3\eta^2-1)\coth^{-1}\eta-3\eta\}]$$

where K is a constant connected with the mass of the prolate spheroid. As before

(25) $$d\nu = r\left[(\frac{\partial\lambda}{\partial r})^2 - (\frac{\partial\lambda}{\partial z})^2\right] dr + 2r\frac{\partial\lambda}{\partial r}\ \frac{\partial\lambda}{\partial z}\ dz$$

where

(26) $$r\left[(\frac{\partial\lambda}{\partial r})^2 - (\frac{\partial\lambda}{\partial z})^2\right] dr$$

$$= K^2\left\{\left[\eta^3(1-\xi^2)(1-5\xi^2)-\eta(1-2\xi^2+\xi^4)\right](\coth^{-1}\eta)^2-\left[2\eta^2(1-\xi^2)(1-5\xi^2)\right]\coth^{-1}\eta\right.$$

$$\left. + \frac{[\eta^3(1-5\xi^2)(1-\xi^2)+4\eta\xi^2(1-\xi^2)]}{(\eta^2-1)}\right\} d\eta -$$

$$- K^2\left\{\left[\eta^2(\eta^2-1)\xi(1-5\xi^2)-\xi(\eta^2-1)(1-\xi^2)\right](\cot^{-1}\eta)^2 - \right.$$

$$\left. - \left[2\xi\eta(1-5\xi^2)(\eta^2-1)\right](\coth^{-1}\eta) + \left[\eta^2\xi(1-5\xi^2)+4\xi^3\right]\right\} d\xi$$

and

$$(27) \qquad 2r \frac{\partial\lambda}{\partial r} \frac{\partial\lambda}{\partial z} \, dz$$

$$= -K^2 \left\{ \left[4\eta\xi^2(\eta^2-1)(1-\xi^2) \right] (\coth^{-1}\eta)^2 + \left[4\xi^2(1-\xi^2)(1-2\eta^2) \right] \coth^{-1}\eta + \left[4\xi^2\eta(1-\xi^2) \right] \right\} d\eta$$

$$-K^2 \left\{ \left[4\xi\eta^2(1-\xi^2)(\eta^2-1) \right] (\coth^{-1}\eta)^2 + \left[4\xi\eta(1-\xi^2)(1-2\eta^2) \right] \coth^{-1}\eta + \left[4\eta^2\xi(1-\xi^2) \right] \right\} d\xi \; .$$

From (26), (27) and (25) we have

$$(28) \qquad d\nu = K^2 \left\{ \left[\eta^3 + 9\eta^3\xi^4 - 10\eta^3\xi^2 + 6\eta\xi^2 - 5\eta\xi^4 - \eta \right] (\coth^{-1}\eta)^2 + \right.$$

$$+ \left[20\xi^2\eta^2 - 18\eta^2\xi^4 - 2\eta^2 + 4\xi^4 - 4\eta^4 \right] (\coth^{-1}\eta) +$$

$$\left. + \frac{[\eta^3 - 10\eta^3\xi^2 + 9\eta^3\xi^4 + 8\eta\xi^2 - 8\eta\xi^4]}{(\eta^2 - 1)} \right\} d\eta \; +$$

$$+ \; K^2 \left\{ \left[9\eta^4\xi^3 - 5\eta^4\xi + 6\eta^2\xi - 10\eta^2\xi^3 + \xi^3 - \xi \right] (\coth^{-1}\eta)^2 + \right.$$

$$\left. + \left[10\eta^3\xi - 6\eta\xi - 18\eta^3\xi^3 + 14\eta\xi^3 \right] (\coth^{-1}\eta) + \left[9\eta^2\xi^3 - 5\eta^2\xi - 4\xi^3 \right] \right\} d\xi$$

Hence

$$(29) \qquad \frac{\partial\nu}{\partial\eta} = K^2\left\{\left[\eta^3(1+9\xi^4-10\xi^2)+\eta(6\xi^2-5\xi^4-1)\right](\coth^{-1}\eta)^2 + \right.$$

$$+ \left[\eta^2(20\xi^2-18\xi^4-2)+4\xi^2(\xi^2-1)\right](\coth^{-1}\eta +$$

$$\left. + \frac{[\eta^3(1+9\xi^4-10\xi^2)+8\xi^2\eta(1-\xi^2)]}{(\eta^2-1)}\right\}$$

and

$$(30) \qquad \frac{\partial\nu}{\partial\xi} = K^2\left\{\left[9\eta^4\xi^3-5\eta^4\xi+6\eta^2\xi-10\eta^2\xi^3+\xi^3-\xi\right](\coth^{-1}\eta)^2\right.$$

$$\left. + \left[10\eta^3\xi-6\xi\eta-18\eta^3\xi^3+14\eta\xi^3\right](\coth^{-1}\eta)+\left[9\eta^2\xi^3-5\eta^2\xi-4\xi^3\right]\right\}.$$

In order to determine the value of ν from (29) , we consider the evaluation of the integral

$$(31) \qquad \int \eta^3(\coth^{-1}\eta)^2\, d\eta = \frac{\eta^4}{4}(\coth^{-1}\eta)^2 + \tfrac{1}{2}\int \frac{\eta^4\coth^{-1}\eta}{(\eta^2-1)}\, d\eta$$

$$= \frac{\eta^4}{4}(\coth^{-1}\eta)^2+\tfrac{1}{2}\int \frac{(\eta^4-\eta^2)+(\eta^2-1)+1}{(\eta^2-1)}\coth^{-1}\eta\, d\eta$$

$$= \frac{\eta^4}{4}(\coth^{-1}\eta)^2 + \tfrac{1}{2}\int \eta^2 \coth^{-1}\eta \, d\eta + \tfrac{1}{2}\int \coth^{-1}\eta \, d\eta + \tfrac{1}{2}\int \frac{\coth^{-1}\eta}{(\eta^2-1)} \, d\eta$$

$$= \frac{\eta^4}{4}(\coth^{-1}\eta)^2 + \tfrac{1}{2}\left[\frac{\eta^3}{3}\coth^{-1}\eta + \frac{1}{3}\int \frac{\eta^3}{(\eta^2-1)} \, d\eta\right] + \tfrac{1}{2}\eta\coth^{-1}\eta + \tfrac{1}{2}\int \frac{\eta d\eta}{\eta^2-1}$$

$$-\tfrac{1}{4}(\coth^{-1}\eta)^2$$

$$= \frac{\eta^4}{4}(\coth^{-1}\eta)^2 + \frac{1}{6}\eta^3 \coth^{-1}\eta + \frac{1}{6}\int \{\eta + \frac{\eta}{\eta^2-1}\} \, d\eta + \frac{\eta}{2}\coth^{-1}\eta + \tfrac{1}{4}\log(\eta^2-1)$$

$$-\tfrac{1}{4}(\coth^{-1}\eta)^2$$

$$= \tfrac{1}{4}(\eta^4-1)(\coth^{-1}\eta)^2 + \frac{1}{6}\eta(\eta^2+3)\coth^{-1}\eta + \frac{1}{3}\log(\eta^2-1) + \frac{\eta^2}{12} \quad .$$

Similarly

(32) $$\int \eta(\coth^{-1}\eta)^2 \, d\eta = \tfrac{1}{2}(\eta^2-1)(\coth^{-1}\eta)^2 + \eta\coth^{-1}\eta + \tfrac{1}{2}\log(\eta^2-1)$$

(33) $$\int \eta^2 \coth^{-1}\eta \, d\eta = \frac{\eta^3}{3}\coth^{-1}\eta + \frac{1}{6}\eta^2 + \frac{1}{6}\log(\eta^2-1)$$

and

(34) $$\int \frac{\eta^3}{(\eta^2-1)} \, d\eta = \frac{\eta^2}{2} + \tfrac{1}{2}\log(\eta^2-1) \quad .$$

From (29) to (34) we have

(35) $$\nu = K^2\Big\{(1-10\xi^2+9\xi^4)\Big[\tfrac{1}{4}(\eta^4-1)(\coth^{-1}\eta)^2+\frac{1}{6}\,\eta(\eta^2+3)\coth^{-1}\eta+\frac{\eta^2}{12}\Big]+$$

$$+\ (6\xi^2-5\xi^4-1)\Big[\tfrac{1}{2}(\eta^2-1)(\coth^{-1}\eta)^2+\eta\coth^{-1}\eta\Big]$$

$$-\ 2(1-10\xi^2+9\xi^4)\Big[\frac{\eta^3}{3}\coth^{-1}\eta+\frac{1}{6}\,\eta^2\Big]-4\xi^2(1-\xi^2)\Big[\eta\coth^{-1}\eta\Big]$$

$$+\ \frac{\eta^2}{2}\Big[1-10\xi^2+9\xi^4\Big]\Big\}+g(\xi)$$

where $g(\xi)$ is an arbitrary function of ξ .

Differentiating (35) partially with respect to ξ we get

(36) $$\frac{\partial\nu}{\partial\xi} = K^2\Big\{\Big[9\eta^4\xi^3-5\eta^4\xi+6\eta^2\xi-10\eta^2\xi^3+\xi^3-\xi\Big](\coth^{-1}\eta)^2\ +$$

$$\Big[10\eta^3\xi-6\xi\eta-18\eta^3\xi^3+14\eta\xi^3\Big](\coth^{-1}\eta)+\Big[9\eta^2\xi^3-5\eta^2\xi\Big]\Big\}+g'(\xi)\ .$$

Comparing (36) with (30) we have

(37) $$g'(\xi) = -4K^2\xi^3\ .$$

Integrating (37) we have

$$g(\xi) = -K^2\xi^4+K^2D\ ,$$ where D is a constant of integration.

From (35) and (37) we have

(38) $$\nu = K^2\Big\{\tfrac{1}{4}(\eta^2-1)\Big[\eta^2(1-10\xi^2+9\xi^4)-(1-\xi^2)^2\Big]\,(\coth^{-1}\eta)^2 -$$

$$-\frac{\eta}{2}\Big[\eta^2(1-10\xi^2+9\xi^4)+(1-6\xi^2+7\xi^4)\Big]\coth^{-1}\eta \;+$$

$$+\Big[\frac{\eta^2}{4}(1-10\xi^2+9\xi^4)-\eta^4+D\Big]\Big\}\;.$$

As before the constant of integration D can be calculated from the condition that the line-element becomes flat at infinity, i.e. $\nu \to 0$ when $r = 0$, $z \to \infty$, corresponding to $\eta \to \infty$, $\xi = 1$. This condition is satisfied if and only if $D = 1$ in (38).

Hence

(39) $$\nu = K^2\Big\{\tfrac{1}{4}(\eta^2-1)\Big[\eta^2(1-10\xi^2+9\xi^4)-(1-\xi^2)^2\Big]\,(\coth^{-1}\eta)^2 -$$

$$-\frac{\eta}{2}\Big[\eta^2(1-10\xi^2+9\xi^4)+(1+6\xi^2-7\xi^4)\Big]\,(\coth^{-1}\eta)\;+$$

$$+\Big[\frac{\eta^2}{4}(1-10\xi^2+9\xi^4)-\eta^4+1\Big]\Big\}$$

or

$$\nu = 9\beta^2\xi^4\Big[(\eta^2-1)(9\eta^2-1)(\coth^{-1}\eta)^2+(14\eta-18\eta^3)\coth^{-1}\eta+9\eta^2-4\Big]$$

$$+\,2\beta^2\xi^2\Big[(\eta^2-1)(1-5\eta^2)(\coth^{-1}\eta)^2+(10\eta^3-6\eta)\coth^{-1}\eta-5\eta^2\Big]$$

$$+\,\beta^2\Big[(\eta^2-1)^2(\coth^{-1}\eta)^2-2(\eta^3+\eta)+\eta^2+4\Big]$$

(where $\beta^2 = 4K^2$)

APPENDIX V

FIELD OF A MASS DIPOLE

Below are the detailed calculations for the expression ν , for the field of a mass dipole (c.f. Chapter III, page 52).

If λ is the Newtonian potential of a dipole, then

(1) $$\lambda = \frac{mz}{\rho^3}$$

where $\rho^2 = r^2 + z^2$.

Levi-Civita equations are

(2)
$$\frac{\partial \nu}{\partial r} = r\left[\left(\frac{\partial \lambda}{\partial r}\right)^2 - \left(\frac{\partial \lambda}{\partial z}\right)^2\right]$$
$$\frac{\partial \nu}{\partial z} = 2r \frac{\partial \lambda}{\partial r} \frac{\partial \lambda}{\partial z} .$$

From (1)

$$\frac{\partial \lambda}{\partial r} = m\,z \frac{\partial}{\partial r}\left(\frac{1}{\rho^3}\right) = mz\left(-\frac{3}{\rho^4} \frac{r}{\rho}\right)$$

(3) $$= -\frac{3mzr}{\rho^5}$$

$$\frac{\partial \lambda}{\partial z} = \frac{m}{\rho^3} - \frac{3mz}{\rho^3} \frac{z}{\rho} = \frac{m}{\rho^3} - \frac{3mz^2}{\rho^5} .$$

Hence

$$\frac{\partial v}{\partial z} = -2r\left(\frac{3mrz}{\rho^5}\right)\left(\frac{m}{\rho^3} - \frac{3mz^2}{\rho^5}\right)$$

$$= -\frac{6m^2r^2z}{\rho^5}\left(\frac{1}{\rho^3} - \frac{3z^2}{\rho^5}\right)$$

or

$$\frac{\partial v}{\partial z} = -\frac{6m^2r^2z}{\rho^8} + \frac{18m^2r^2z^3}{\rho^{10}}$$

$$= -\frac{6m^2r^2z}{\rho^8} + \frac{18m^2r^2z(\rho^2-r^2)}{\rho^{10}} \quad (\text{as } \rho^2 = r^2+z^2) .$$

(4)
$$\frac{\partial v}{\partial z} = \frac{12m^2r^2z}{\rho^8} - \frac{18m^2r^4z}{\rho^{10}} .$$

Integrating we have

(5)
$$v = -\frac{2m^2r^2}{\rho^6} + \frac{9}{4}\frac{m^2r^4}{\rho^8} + f(r) .$$

Differentiating (5) partially w. r. to r.

$$\frac{\partial v}{\partial r} = -\frac{4m^2r}{\rho^6} + \frac{12m^2r^3}{\rho^8} + \frac{9m^2r^3}{\rho^8} - \frac{18m^2r^5}{\rho^{10}} + f'(r)$$

(6)
$$= -\frac{4m^2r}{\rho^6} + \frac{21m^2r^3}{\rho^8} - \frac{18m^2r^5}{\rho^{10}} + f'(r) .$$

Also from (2) and (3)

$$(7) \qquad \frac{\partial \nu}{\partial r} = \frac{9m^2r^3z^2}{\rho^{10}} - \frac{m^2r}{\rho^6} - \frac{9m^2rz^4}{\rho^{10}} + \frac{6m^2rz^2}{\rho^8} .$$

Comparing (6) and (7)

$$f'(r) = \frac{9m^2r^3z^2}{\rho^{10}} - \frac{m^2r}{\rho^6} - \frac{9m^2rz^4}{\rho^{10}} + \frac{6m^2rz^2}{\rho^8} + \frac{4m^2r}{\rho^6} - \frac{21m^2r^3}{\rho^8} + \frac{18m^2r^5}{\rho^{10}}$$

$$f'(r) = \frac{9m^2r^3z^2}{\rho^{10}} + \frac{3m^2r}{\rho^6} - \frac{9m^2rz^4}{\rho^{10}} + \frac{6m^2rz^2}{\rho^8} - \frac{21m^2r^3}{\rho^8} + \frac{18m^2r^5}{\rho^{10}}$$

$$= \frac{3m^2r}{\rho^{10}} [3r^2z^2 + \rho^4 - 3z^4 + 2\rho^2z^2 - 7r^2\rho^2 + 6r^4]$$

$$= \frac{3m^2r}{\rho^{10}} [3r^2z^2 + (r^2+z^2)^2 - 3z^4 + 2(r^2+z^2)z^2 - 7r^2(r^2+z^2)+6r^4]$$

$$= \frac{3m^2r}{\rho^{10}} [3r^2z^2+r^4+2r^2z^2+z^4-3z^4+2r^2z^2+2z^4-7r^4-7r^2z^2+6r^4]$$

$$= 0 .$$

Hence from (1) and (5)

$$\lambda = \frac{mz}{\rho^3}$$

$$\nu = -\frac{2m^2r^2}{\rho^6} + \frac{9}{4}\frac{m^2r^4}{\rho^8} .$$

APPENDIX VI

DERIVATION OF FIELD OF n-BODIES.

Following are the detailed calculations for the function ν_{ij} for n collinear axially symmetric bodies (c.f. Chapter III, page 55).

In the case of a Weyl manifold whose Newtonian analogue is the static field of n distinct rods $B_1, B_2, B_3, \dots B_n$, we set

$$(1) \qquad \lambda = \sum_{i=1}^{n} \lambda_i$$

where

$$(2) \qquad \lambda_i = \frac{m_i}{b_i} \log \frac{\rho_i + \rho_i' - b_i}{\rho_i + \rho_i' + b_i}$$

is the external Newtonian potential of B_i of mass m_i extending from $z = a_i - \frac{1}{2}b_i$ to $z = a_i + \frac{1}{2}b_i$ and thus satisfies

$$(3) \qquad \nabla^2 \lambda_i = 0$$

$$(4) \qquad \rho_i^2 = r^2 + z_i^2 \ , \quad \rho_i'^2 = r^2 + z_i'^2$$

$$(5) \qquad z_i = z - (a_i + \tfrac{1}{2}b_i) \ , \ z_i' = z - (a_i - \tfrac{1}{2}b_i) \ .$$

To obtain ν from (3.2) (Chapter III), we compute ν_{ij} from

(6) $$\frac{\partial \nu_{ij}}{\partial r} = r\left(\frac{\partial \lambda_i}{\partial r}\frac{\partial \lambda_j}{\partial r} - \frac{\partial \lambda_i}{\partial z}\frac{\partial \lambda_j}{\partial z}\right)$$

(7) $$\frac{\partial \nu_{ij}}{\partial z} = r\left(\frac{\partial \lambda_i}{\partial r}\frac{\partial \lambda_j}{\partial z} + \frac{\partial \lambda_i}{\partial z}\frac{\partial \lambda_j}{\partial r}\right)$$

we have

(8) $$z - a_i = \tfrac{1}{2}b_i\,\xi_i\,\eta_i\ , \quad r = \tfrac{1}{2}b_i(\eta_i^2 - 1)^{\frac{1}{2}}(1-\xi_i^2)^{\frac{1}{2}}$$

where

$$\eta_i = \frac{\rho_i + \rho'_i}{b_i} \quad \text{and} \quad \xi_i = \frac{\rho'_i - \rho_i}{b_i} \quad .$$

Similarly,

(9) $$z-a_j = \tfrac{1}{2}b_j\,\xi_j\,\eta_j\ , \quad r = \tfrac{1}{2}b_j(\eta_j^2 - 1)^{\frac{1}{2}}(1-\xi_j^2)^{\frac{1}{2}}$$

where

$$\eta_j = \frac{\rho_j + \rho'_j}{b_j}\ , \quad \xi_j = \frac{\rho'_j - \rho_j}{b_j} \quad .$$

From (8) and (2) we have

(10) $$\lambda_i = \frac{m_i}{b_i}\log\frac{\eta_i - 1}{\eta_i + 1}$$

also

(11) $$\frac{\partial \eta_i}{\partial r} = \frac{2\eta_i(\eta_i^2 - 1)^{\frac{1}{2}}(1-\xi_i^2)^{\frac{1}{2}}}{b_i(\eta_i^2 - \xi_i^2)}\ , \quad \frac{\partial \eta_i}{\partial z} = \frac{2\xi(\eta_i^2 - 1)}{b_i(\eta_i^2 - \xi_i^2)}$$

$$\frac{\partial \xi_i}{\partial r} = -\frac{2\xi_i(\eta_i^2 - 1)^{\frac{1}{2}}(1-\xi_i^2)^{\frac{1}{2}}}{b_i(\eta_i^2 - \xi_i^2)}\ , \quad \frac{\partial \xi_i}{\partial z} = \frac{2\eta_i(1-\xi_i^2)}{b_i(\eta_i^2 - \xi_i^2)} \quad .$$

From (10) we have

$$
(12)\qquad
\begin{aligned}
\frac{\partial\lambda_i}{\partial r} &= \frac{2m_i}{b_i}\,\frac{1}{(\eta_i^2-1)}\,\frac{\partial\eta_i}{\partial r} = \frac{4m_i}{b_i}\,\frac{\eta_i(1-\xi_i^2)^{\frac{1}{2}}}{(\eta_i^2-\xi_i^2)(\eta_i^2-1)^{\frac{1}{2}}}\\
\frac{\partial\lambda_i}{\partial z} &= \frac{2m_i}{b_i}\,\frac{1}{(\eta_i^2-1)}\,\frac{\partial\eta_i}{\partial z} = \frac{4m_i}{b_i}\,\frac{4m_i\xi_i}{b_i(\eta_i^2-\xi_i^2)} \quad .
\end{aligned}
$$

Hence

$$
(13)\qquad \left(\frac{\partial\lambda_i}{\partial r}\,\frac{\partial\lambda_j}{\partial z} + \frac{\partial\lambda_i}{\partial z}\,\frac{\partial\lambda_j}{\partial z}\right)
$$

$$
= \frac{4m_i}{b_i}\,\frac{\eta_i(1-\xi_i^2)^{\frac{1}{2}}}{(\eta_i^2-\xi_i^2)(\eta_i^2-1)^{\frac{1}{2}}}\,\frac{4m_j}{b_j}\,\frac{\xi_i}{(\eta_j^2-\xi_j^2)} + \frac{4m_j}{b_j}\,\frac{\eta_j(1-\xi_j)^{\frac{1}{2}}}{(\eta_j^2-\xi_j^2)(\eta_j^2-1)^{\frac{1}{2}}}
$$

$$
\frac{4m_i}{b_i}\,\frac{\xi_i}{(\eta_i^2-\xi_i^2)} \quad .
$$

Also from (8) and (9)

$$
(14)\qquad
\begin{aligned}
(\eta_i^2-1)^{\frac{1}{2}} &= \frac{2r}{b_i(1-\xi_i^2)^{\frac{1}{2}}}\\
(\eta_j^2-1)^{\frac{1}{2}} &= \frac{2r}{b_j(1-\xi_j^2)^{\frac{1}{2}}} \quad .
\end{aligned}
$$

Hence from (7), (13) and (14) we have

$$(15)\; z\,\frac{\partial \nu_{ij}}{\partial z} = 2r\left(\frac{\partial \lambda_i}{\partial r}\,\frac{\partial \lambda_j}{\partial z} + \frac{\partial \lambda_i}{\partial z}\,\frac{\partial \lambda_j}{\partial r} \right)$$

$$= \frac{16 m_i m_j \eta_i \xi_j (1-\xi_i^2)}{(b_i b_j) b_j (\eta_i^2-\xi_i^2)(\xi_j^2-\xi_j^2)} + \frac{16 m_i m_j \eta_j \xi_i (1-\xi_j^2)}{(b_i b_j) b_i (\eta_i^2-\xi_i^2)(\eta_j^2-\xi_j^2}$$

$$= \frac{16 m_i m_j}{b_i b_j (\eta_i^2-\xi_i^2)(\eta_j^2-\xi_j^2)} \left[\frac{\eta_i \xi_j}{b_j} (1-\xi_i^2) + \frac{\eta_j \xi_i}{b_i} (1-\xi_j^2) \right]$$

$$\begin{aligned} = \frac{16 m_i m_j}{b_i b_j (\eta_i^2-\xi_i^2)(\eta_j^2-\xi_j^2)} \Bigg[& \frac{1}{b_j} \left\{ \frac{\rho_i+\rho_i'}{b_i}\,\frac{\rho_j'-\rho_j}{b_j} - \frac{2(z-a_i)}{b_i}\,\frac{\rho_i'-\rho_i}{b_i}\,\frac{\rho_j'-\rho_j}{b_j} \right\} \\ & + \frac{1}{b_i} \left\{ \frac{\rho_j+\rho_j'}{b_j}\,\frac{\rho_i'-\rho_i}{b_i} - \frac{2(z-a_j)}{b_j}\,\frac{\rho_j'-\rho_j}{b_j}\,\frac{\rho_i'-\rho_i}{b_i} \right\} \Bigg] \end{aligned}$$

$$\begin{aligned} = \frac{16 m_i m_j}{b_i b_j (\eta_i^2-\xi_i^2)(\eta_j^2-\xi_j^2)} \Bigg[& \frac{(\rho_i+\rho_i')(\rho_j-\rho_j)}{b_j^2 b_i} - \frac{2(z-a_i)(\rho_i'-\rho_i)(\rho_j'-\rho_j)}{b_i^2\, b_j^2} \\ & + \frac{(\rho_j+\rho_j'(\rho_i'-\rho_i)}{b_i^2 b_j} - \frac{2(z-a_j)(\rho_j'-\rho_j)(\rho_i'-\rho_i)}{b_i^2\, b_j^2} \Bigg] \end{aligned}$$

$$\begin{aligned} = \frac{b_i b_j m_i m_j}{\rho_i \rho_j \rho_i' \rho_j'} \Bigg[& \frac{(\rho_i+\rho_i')(\rho_j'-\rho_j)}{b_i\, b_j^2} - \frac{2(z-a_j)(\rho_i'-\rho_i)(\rho_j'-\rho_j)}{b_i^2\, b_j^2} \\ & + \frac{(\rho_j+\rho_j'(\rho_i'-\rho_i)}{b_i^2\, b_j} - \frac{2(z-a_j)(\rho_j'-\rho_j)(\rho_i'-\rho_i)}{b_i^2\, b_j^2} \Bigg] . \end{aligned}$$

$$\frac{\partial v_{ij}}{\partial z} = \frac{(m_i m_j/b_i b_j)}{\rho_i\ \rho_j\ \rho_i'\ \rho_j'}\Big[b_i\{\rho_i\rho_j' - \rho_i\rho_j + \rho_i'\rho_j' - \rho_i'\rho_j\} - 2(z-a_i)\{\rho_i\rho_j - \rho_i'\rho_j - \rho_i\rho_j' + \rho_i'\rho_j'\}$$

$$+ b_j\{\rho_j\rho_i' - \rho_i\rho_j + \rho_i'\rho_j' - \rho_i\rho_j'\} - 2(z-a_j)\{\rho_i'\rho_j' - \rho_i\rho_j' - \rho_i'\rho_j + \rho_i\rho_j\}\Big]$$

$$= \frac{(m_i m_j/b_i b_j)}{\rho_i\ \rho_j\ \rho_i'\ \rho_j'}\Big[-\rho_i\rho_j\ \{b_i + 2(z-a_i) + b_j + 2(z-a_j)\}$$

$$+\rho_i\rho_j'\ \{b_i + 2(z-a_i) - b_j + 2(z-a_j)\}$$

$$-\rho_i'\rho_j\ \{b_i - 2(z-a_i) - b_j - 2(z-a_j)\}$$

$$+\rho_i'\rho_j'\ \{b_i - 2(z-a_i) + b_j - 2(z-a_j)\}\Big]$$

$$= \frac{(m_i m_j/b_i b_j)}{\rho_i\ \rho_j\ \rho_i'\ \rho_j'}\Big[-\rho_i\rho_j\{2z_i' + 2z_j'\} + \rho_i\rho_j'\{2z_i' + 2z_j\}$$

$$-\rho_i'\rho_j\{-2z_i - 2z_j'\} + \rho_i'\rho_j'\{-2z_i - 2z_j\}\Big] .$$

$$(16)\qquad 2\ \frac{\partial v_{ij}}{\partial z} = \frac{2m_i m_j}{b_i b_j}\left[-\frac{(z_i'+z_j')}{\rho_i'\ \rho_j'} + \frac{(z_i'+z_j)}{\rho_i'\ \rho_j} + \frac{(z_i+z_j')}{\rho_i\ \rho_j'} - \frac{(z_i+z_j)}{\rho_i'\ \rho_j'}\right].$$

By straightforward but tedious calculations using (6), (8) and (9), we find that

$$(17)\qquad 2\ \frac{\partial v_{ij}}{\partial r} = \frac{2m_i m_j}{b_i b_j}\left[-\frac{z_i'z_j'-r^2}{r\rho_i'\rho_j'} + \frac{z_i'z_j-r^2}{r\rho_i'\rho_j} + \frac{z_iz_j'-r^2}{r\rho_i\rho_j'} - \frac{z_iz_j-r^2}{r\rho_i\rho_j}\right] .$$

In order to get the value of ν_{ij} from (16) and (17), let us establish the following identity

(18) $$d\ \log[r^{-1}E(i',j)] = \frac{r^2 - z_i'z_j}{r\ \rho_i'\rho_j}\,dr + \frac{z_i'+z_j}{\rho_i\rho_j}\,dz \ .$$

Set

(19) $$f = \log[r^{-1}E(i',j)] = \log\left[\frac{\rho_i'\rho_j + z_i'z_j + r^2}{r}\right]$$

then

$$\frac{\partial f}{\partial r} = \frac{r}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{r\{\rho_i'\frac{\partial\rho_j}{\partial r} + \rho_j\frac{\partial\rho_i'}{\partial r} + 2r\} - \{\rho_i'\rho_j+z_i'z_j+r^2\}}{r^2}\right]$$

$$= \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{r\{\rho_i'\frac{r}{\rho_j} + \rho_j\frac{r}{\rho_i'} + 2r\} - \{\rho_i'\rho_j+z_i'z_j+r^2\}}{r}\right]$$

$$= \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{r^2\rho_i'^2+r^2\rho_j^2+2r^2\rho_i'\rho_j-\rho_i'^2\rho_j^2-z_i'z_j\rho_i'\rho_j-r^2\rho_i'\rho_j}{r\rho_i'\rho_j}\right]$$

$$= \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{r^2(r^2+z_i'^2)+r^2(r^2+z_j^2)+r^2\rho_i'\rho_j-(r^2+z_i'^2)(r^2+z^2)-z_i'z_j\rho_i'\rho_j}{r\ \rho_i'\ \rho_j}\right]$$

$$= \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{r^4 + r^2\rho_i'\rho_j - z_i'^2z_j^2 - z_i'z_j\rho_i'\rho_j}{r\ \rho_i'\ \rho_j}\right]$$

$$\frac{\partial f}{\partial r} = \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{(r^2-z_i'z_j)(\rho_i'\rho_j + z_i'z_j + r^2)}{r\,\rho_i'\,\rho_j}\right]$$

$$= \frac{r^2 - z_i'z_j}{r\,\rho_i'\,\rho_j}$$

i.e.

$$\frac{\partial}{\partial r}\log[r^{-1}E(i',j)] = \frac{r^2 - z_i'z_j}{r\,\rho_i'\,\rho_j} \quad . \tag{20}$$

Again from (19) we have

$$\frac{\partial f}{\partial z} = \frac{r}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{r\{\rho_i'\frac{\partial\rho_j}{\partial z} + \rho_j\frac{\partial\rho_i'}{\partial z} + z_i'\frac{\partial z_j}{\partial z} + z_j\frac{\partial z_i'}{\partial z}\}}{r^2}\right]$$

$$= \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\rho_i'\frac{z_j}{\rho_j} + \rho_j\frac{z_i'}{\rho_i'} + z_i' + z_j\right]$$

$$= \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{\rho_i'^2 z_j + z_i'\rho_j^2 + z_i'\rho_i'\rho_j + z_j\rho_i'\rho_j}{\rho_i'\,\rho_j}\right]$$

$$= \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)}\left[\frac{(r^2+z_i'^2)z_j + z_i'(r^2+z_j^2) + z_i'\rho_i'\rho_j + z_j\rho_i'\rho_j}{\rho_i'\,\rho_j}\right]$$

$$= \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)} \left[\frac{r^2(z_i'+z_j)+z_i'z_j(z_i'+z_j)+\rho_i'\rho_j(z_i'+z_j)}{\rho_i'\,\rho_j} \right]$$

or

$$\frac{\partial f}{\partial z} = \frac{1}{(\rho_i'\rho_j+z_i'z_j+r^2)} \quad \frac{(z_i' + z_j)(\rho_i'\rho_j + z_i'z_j + r^2)}{\rho_i'\,\rho_j}$$

i.e.

$$\text{(21)} \qquad \frac{\partial}{\partial z} \log[r^{-1}E(i',j)] = \frac{z_i' + z_j}{\rho_i'\,\rho_j} \quad .$$

Hence the identity (18) is proved.

Exactly the same way we can show that

$$d \log[r^{-1}E(i,j')] = \frac{r^2 - z_i z_j'}{r\,\rho_i\rho_j'}\,dr + \frac{z_i + z_j'}{\rho_i\rho_j'}\,dz$$

$$\text{(22)} \qquad d \log[r^{-1}E(i,j)] = \frac{r^2 - z_i z_j}{r\,\rho_i\rho_j}\,dr + \frac{z_i + z_j}{\rho_i\rho_j}\,dz$$

$$d \log[r^{-1}E(i',j')] = \frac{r^2 - z_i'z_j'}{r\,\rho_i'\rho_j'}\,dr + \frac{z_i' + z_j'}{\rho_i'\rho_j}\,dz \quad .$$

From (16) to (22) we have

$$\text{(23)} \qquad \mathcal{Z}\,d\nu_{ij} = \frac{2m_i m_j}{b_i b_j} \Big[d \log\{r^{-1}E(i',j)\} + d \log\{r^{-1}E(i,j')\} - d \log\{r^{-1}E(i,j)\} - d \log\{r^{-1}E(i',j')\} \Big]$$

Hence

$$(24) \qquad 2\nu_{ij} = \frac{2m_i m_j}{b_i b_j} \log \left[\frac{E(i',j)\ E(i,j')}{E(i,j)\ E(i',j')} \right] .$$

APPENDIX VII

LIMITING PROCESS OF SCHWARZSCHILD PARTICLES.

In this appendix we show that when each $b_i \to 0$, the solution (3.61) is formally the field of n collinear Curzon particles (c.f. Chapter III, p.55).

From equation (3.56), Chapter III, we have

(1) $$\lambda_i = \frac{m_i}{b_i} \log \frac{\rho_i + \rho_i' - b_i}{\rho_i + \rho_i' + b_i}$$

where

(2) $$\rho_i^2 = r^2 + z_i^2 , \quad \rho_i'^2 = r^2 + z_i'^2 , \quad m_i \text{ is constant}$$

(3) $$z_i = z - (a_i + \tfrac{1}{2}b_i), \quad z_i' = z - (a_i - \tfrac{1}{2}b_i) .$$

In (1) set

(4) $$f(\rho_i, \rho_i', b_i) = m_i \log \frac{\rho_i + \rho_i' - b_i}{\rho_i + \rho_i' + b_i}$$

and

(5) $$g(b_i) = b_i .$$

When $b_i \to 0$, $f(\rho_i, \rho_i', b_i) \to 0$, $g(b_i) \to 0$. So λ_i is indeterminate when $b_i = 0$.

Using L'Hospital's rule,

$$(6) \qquad \frac{\partial f}{\partial b_i} = m_i \, \frac{\rho_i+\rho_i'+b_i}{\rho_i+\rho_i'-b_i} \; \frac{(\rho_i+\rho_i'+b_i)\{\frac{\partial \rho_i}{\partial b_i} + \frac{\partial \rho_i'}{\partial b_i} -1\} - (\rho_i+\rho_i'-b_i)\{\frac{\partial \rho_i}{\partial b_i} + \frac{\partial \rho_i}{\partial b_i} +1\}}{(\rho_i + \rho_i' + b_i)^2}$$

From (2) and (3)

$$(7) \qquad \frac{\partial \rho_i}{\partial b_i} = -\frac{z_i}{2\rho_i} \, , \qquad \frac{\partial \rho_i'}{\partial b_i} = \frac{z_i'}{2\rho_i'} \; .$$

From (7) and (6), we have

$$\frac{\partial f}{\partial b_i} = m_i \frac{1}{(\rho_i+\rho_i'-b_i)(\rho_i+\rho_i'+b_i)} \Big\{ (\rho_i+\rho_i'+b_i)(\frac{-z_i}{2\rho_i} + \frac{z_i'}{2\rho_i'} -1)$$

$$-(\rho_i+\rho_i'-b_i)(\frac{-z_i}{2\rho_i} + \frac{z_i'}{2\rho_i} + 1) \Big\}$$

$$= \frac{m_i}{(\rho_i+\rho_i'-b_i)(\rho_i+\rho_i'+b_i)} \Big\{ b_i \, (\frac{z_i'}{\rho_i'} - \frac{z_i}{\rho_i}) - 2(\rho_i + \rho_i') \Big\}$$

$$\lim_{b_i \to o} \frac{\partial f}{\partial b_i} = -\frac{2m_i(\rho_i)}{2\rho_i^2} \qquad (\text{since } \rho_i' \to \rho_i \text{ when } b_i \to o)$$

$$= -\frac{m_i}{\rho_i} \; .$$

Also

$$(9) \qquad \lim_{b_i \to o} \frac{\partial g}{\partial b_i} = 1 \; .$$

From (8) and (9)

$$\lambda_i = -\frac{m_i}{\rho_i} \tag{10}$$

which is the Newtonian potential for n "Curzon particles".

Also from equation (3.61), Chapter III, we have

$$\begin{aligned} \nu_{ij} &= \frac{m_i m_j}{b_i b_j} \log \frac{E(i',j)E(i,j')}{E(i,j)E(i',j')} \\ &= \frac{m_i m_j}{b_i b_j} [\log E(i',j) + \log E(i,j') - \log E(i,j) - \log E(i',j')] \end{aligned} \tag{11}$$

where

$$\begin{aligned} E(i',j) &= \rho_i'\rho_j + z_i'z_j + r^2 \\ E(i,j') &= \rho_i\rho_j' + z_iz_j' + r^2 \\ E(i,j) &= \rho_i\rho_j + z_iz_j + r^2 \\ \text{and} \quad E(i',j') &= \rho_i'\rho_j' + z_i'z_j' + r^2 \quad . \end{aligned} \tag{12}$$

In (11), set

$$\begin{aligned} \varphi(\rho,z,b) &= m_i m_j \log \frac{E(i'j)E(i,j')}{E(i,j)E(i',j')} \\ \psi(b) &= b_i b_j \quad . \end{aligned} \tag{13}$$

In (11) if we keep b_j fixed and let $b_i \to 0$, then $\varphi \to o$ and $\psi \to 0$ and ν_{ij} indeterminate. As before using L'Hospital's rule of differentiation

we have

$$(14)\qquad \frac{\partial\varphi}{\partial b_i} = m_i m_j \left[\frac{1}{E(i',j)}\,\partial_{b_i} E(i',j) + \frac{1}{E(i,j')}\,\partial_{b_i} E(i,j') - \frac{1}{E(i,j)}\,\partial_{b_i} E(i,j) - \frac{1}{E(i',j')}\,\partial_{b_i} E(i',j')\right].$$

From (7) and (12) we have

$$(15)\qquad \begin{aligned} \partial_{b_i} E(i',j) &= \frac{\rho_j z_i' + \rho_i' z_j}{2\rho_i'} \;; \quad \partial_{b_i} E(i,j') = -\frac{\rho_j' z_i + \rho_i z_j'}{2\rho_i} \\ \partial_{b_i} E(i,j) &= -\frac{\rho_i z_j + \rho_j z_i}{2\rho_i} \;; \quad \partial_{b_i} E(i',j') = \frac{\rho_j' z_i' + \rho_i' z_j'}{2\rho_i'} . \end{aligned}$$

Similarly,

$$(16)\qquad \begin{aligned} \partial_{b_j} E(i',j) &= -\frac{\rho_i' z_j + z_i' \rho_j}{2\rho_j} \;; \quad \partial_{b_j} E(i,j') = \frac{\rho_i z_j' + z_i \rho_j'}{2\rho_j'} \\ \partial_{b_j} E(i,j) &= -\frac{\rho_i z_j + \rho_j z_i}{2\rho_j} \;; \quad \partial_{b_j} E(i',j') = \frac{\rho_i' z_i' + \rho_j' z_i'}{2\rho_j'} \end{aligned}$$

From (14) and (15)

$$(17)\qquad \lim_{b_i \to o} \frac{\partial\varphi}{\partial b_i} = \frac{m_i m_j}{\rho_i}\left[\frac{1}{E(i,j)}\,(\rho_j z_i + z_j \rho_i) - \frac{1}{E(i,j')}\,(\rho_j' z_i + \rho_i z_j')\right]$$

(since $b_i \to o$, $E(i',j) = E(i,j)$; $E(i,j') = E(i',j')$).

Also

$$(18)\qquad \lim_{b_i \to o} \frac{\partial\psi}{\partial b_i} = b_j .$$

Hence,

$$\lim_{b_i \to o} \nu_{ij} = \lim_{b_i \to o} \frac{\frac{\partial \varphi}{\partial b_i}}{\frac{\partial \psi}{\partial b_i}} = \frac{m_i m_j}{b_j \rho_i} \left[\frac{(\rho_j z_i + \rho_i z_j)}{E(i,j)} - \frac{(\rho_j' z_i + \rho_i z_j')}{E(i,j')} \right] . \tag{19}$$

In (19) if $b_j \to o$, then ν_{ij} again becomes indeterminate.

Hence, differentiating the numerator and denominator with respect to b_j we have

$$\nu_{ij} = \frac{m_i m_j}{\rho_i \rho_j (\rho_i \rho_j + z_i z_j + r^2)^2} [-(\rho_i \rho_j + z_i z_j)(\rho_i \rho_j + z_i z_j + r^2) + (z_i \rho_j + z_j \rho_i)^2]$$

$$= - \frac{m_i m_j}{\rho_i \rho_j} \quad \frac{r^2}{(\rho_i \rho_j + z_i z_j + r^2)}$$

$$= - \frac{m_i m_j}{\rho_i \rho_j} \quad \frac{r^2(-\rho_i \rho_j + z_i z_j + r^2)}{(\rho_i \rho_j + z_i z_j + r^2)(r^2 + z_i z_j - \rho_i \rho_j)}$$

$$= - \frac{m_i m_j}{\rho_i \rho_j} \quad \frac{r^2(r^2 + z_i z_j - \rho_i \rho_j)}{(r^2 + z_i z_j)^2 - \rho_i^2 \rho_j^2}$$

$$= - \frac{m_i m_j}{\rho_i \rho_j} \quad \frac{r^2(r^2 + z_i z_j - \rho_i \rho_j)}{(r^2 + z_i z_j)^2 - (r^2 + z_i^2)(r^2 + z_j^2)}$$

$$= \frac{m_i m_j}{\rho_i \rho_j} \quad \frac{(r^2 + z_i z_j - \rho_i \rho_j)}{(z_i - z_j)^2} \quad .$$

APPENDIX VIII

DETERMINATION OF ν_{ij} BETWEEN THE TWO RODS ON THE AXIS OF SYMMETRY.

In this appendix we determine the value of ν_{ij} between i^{th} and j^{th} rods on the axis of symmetry (c.f. Chapter III, page 58).

From equation (3.61), we have

$$\nu_{ij} = \frac{m_i m_j}{b_i b_j} \log \frac{E(i',j)E(i,j')}{E(i,j)E(i',j')} \tag{1}$$

$$= \frac{m_i m_j}{b_i b_j} \log \frac{[r^2 + z_i' z_j + \rho_i' \rho_j][r^2 + z_i z_j' + \rho_i \rho_j']}{[r^2 + z_i z_j + \rho_i \rho_j][r^2 + z_i' z_j' + \rho_i' \rho_j']}$$

where

$$\rho_i^2 = r^2 + z_i^2 \; ; \; \rho_i'^2 = r^2 + z_i'^2 \tag{2}$$

and

$$\begin{aligned} z_i &= z - (a_i + \tfrac{1}{2}b_i) \; ; \; z_i' = z - (a_i - \tfrac{1}{2}b_i) \\ z_j &= z - (a_j + \tfrac{1}{2}b_j) \; ; \; z_j' = z - (a_j - \tfrac{1}{2}b_j) \end{aligned} \tag{3}$$

When $z < a_i - \frac{1}{2}b_i$ on the axis of symmetry, z_i ; z_i' ; z_j and z_j' are all negative and

$$\rho_i' \rho_j = z_i' z_j \; ; \; \rho_i \rho_j' = z_i z_j' \; ; \; \rho_i \rho_j = z_i z_j \text{ and } \rho_i' \rho_j' = z_i' z_j' \, . \tag{4}$$

Hence from (1) and (4)

$$\nu_{ij} = 0 \quad (r = 0,\ i \leq j\ ,\ z < (a_i - \tfrac{1}{2}b_i))\ .$$

Similarly, it can be shown that when $z > a_j + \frac{1}{2}b_j$ then z_i, z_i' , z_j and z_j' are all positive and

$$\nu_{ij} = 0 \quad (r = 0,\ i \leq j,\ z > a_j + \tfrac{1}{2}b_j)\ .$$

Let us now consider the case when z lies between the i^{th} and j^{th} rods.

For all values of z between the i^{th} and the j^{th} rods on the z-axis, z_i and z_i' are always positive, while z_j and z_j' are both negative. ν_{ij} does not vanish in between the i^{th} and the j^{th} rods on the z-axis. From (1) and (2)

$$(5) \qquad (r^2 + z_i'z_j + \rho_i'\rho_j) = [r^2 + z_i'z_j + \sqrt{(r^2+z_i'^2)(r^2+z_j^2)}]$$

$$= [r^2 + z_i'z_j + \sqrt{z_i'^2 z_j^2 + r^2(z_i'^2 + z_j^2) + r^4}]$$

$$= \left[r^2 + z_i'z_j - z_i'z_j \left\{1 + r^2 \frac{(z_i'^2 + z_j^2)}{z_i'^2 z_j^2} + \frac{r^4}{z_i'^2 z_j^2}\right\}^{\frac{1}{2}}\right]$$

$$= \left[r^2 + z_i'z_j - z_i'z_j \left\{1 + \frac{r^2}{2} \quad \frac{z_i'^2 + z_j^2}{z_i'^2 z_j^2} + \ldots\right\}\right]$$

(neglecting higher powers of r^2)

$$= - \frac{r^2}{2} \quad \frac{(z_i' - z_j)^2}{z_i'z_j}\ .$$

Similarly it can be shown that

$$(r^2 + z_i z_j' + \rho_i \rho_j') = -\frac{r^2}{2} \frac{(z_i - z_j')^2}{z_i \; z_j'}$$

(6) $$(r^2 + z_i z_j + \rho_i \rho_j) = -\frac{r^2}{2} \frac{(z_i - z_j)^2}{z_i \; z_j}$$

$$(r^2 + z_i' z_j' + \rho_i' \rho_j') = -\frac{r^2}{2} \frac{(z_i' - z_j')^2}{z_i' \; z_j'} \quad .$$

Hence from (1), (5) and (6)

(7) $$\nu_{ij} = \frac{m_i m_j}{b_i b_j} \log \frac{(z_i' - z_j)^2 (z_i - z_j')^2}{(z_i - z_j)^2 (z_i' - z_j')^2}$$

$$= \frac{2m_i m_j}{b_i b_j} \log \left| \frac{(z_i' - z_j)(z_i - z_j')}{(z_i - z_j)(z_i' - z_j')} \right|$$

$$(r=0, \; i < j, \; a_i + \tfrac{1}{2} b_i < z < a_j - \tfrac{1}{2} b_j) \quad .$$

APPENDIX IX

EQUIVALENCE OF ν_{ij} AND THE NEWTONIAN FORCE BETWEEN THE RODS.

In this appendix we show the equivalence of the Newtonian force between the i^{th} and j^{th} collinear rods separated by the region of the z-axis in the Euclidean map and the relativistic expression ν_{ij} in between these two rods on the axis of symmetry (c.f. Chapter III, page 58).

Let us consider any small length dx_i on the i^{th} rod of length b_i and of uniform mass m_i. Let the distance of this length from the centre of the rod be x_i. Similarly dx_j is any small length on the j^{th} rod of length b_j and of mass uniform m_j. x_j denotes the distance of dx_j from the centre of the j^{th} rod.

If F_{ij} is the force acting between the two rods, then

(1) $$F_{ij} = \frac{m_i m_j}{b_i b_j} \int_{-\frac{1}{2}b_i}^{+\frac{1}{2}b_i} \int_{-\frac{1}{2}b_j}^{+\frac{1}{2}b_j} \frac{dx_i \, dx_j}{(r_{ij} + x_j - x_i)^2} \qquad i < j$$

where r_{ij} is the distance between the centres of the two rods.

On integration, we have

$$F_{ij} = \frac{m_i m_j}{b_i b_j} \int_{-\frac{1}{2}b_i}^{+\frac{1}{2}b_i} \left[- \frac{dx_i}{(r_{ij} + x_j - x_i)} \right]_{-\frac{1}{2}b_j}^{+\frac{1}{2}b_j}$$

$$= \frac{m_i m_j}{b_i b_j} \int_{-\frac{1}{2}b_i}^{+\frac{1}{2}b_i} \left[\frac{dx_i}{(r_{ij} - \frac{1}{2}b_j - x_i)} - \frac{dx_i}{(r_{ij} + \frac{1}{2}b_j - x_i)} \right]$$

$$(2) \qquad F_{ij} = \frac{m_i m_j}{b_i b_j} \log \left| \frac{(r_{ij} + \frac{1}{2}b_j - \frac{1}{2}b_i)(r_{ij} - \frac{1}{2}b_j + \frac{1}{2}b_i)}{(r_{ij} + \frac{1}{2}b_j + \frac{1}{2}b_i)(r_{ij} - \frac{1}{2}b_j - \frac{1}{2}b_i)} \right|$$

which is the same result as we have obtained for the expression $\frac{1}{2}V_{ij}$ between the two i^{th} and j^{th} rod in Appendix VIII, equation (7) .

APPENDIX X

VERIFICATION OF EQUATION (5.18).

Using the Kruskal's formula for the co-ordinate transformation, following are the detailed calculations for the metric transformation (5.18) (c.f. Chapter V, page 97) .

From equation (5.15) Chapter V, we have

(1) $$\varphi = e^{2(\delta-\psi+k)}\left[(1-\frac{b}{R})^{-1}\ (dR)^2 + R^2\ d\theta^2\right] +$$

$$+\ e^{-2\psi+2k}\ R^2\ \sin^2\theta\ d\varphi^2 - e^{2(\psi-k)}(1-\frac{b}{R})\ dT^2 \quad .$$

Defining p, h, ζ and τ by the Kruskal's formula (5.4) we have

(2) $$R = b(1+p) \quad ; \quad h^2 = 4pe^p \quad ,$$

$$\zeta = bh\cosh\frac{T}{2b} \quad ; \quad \tau = bh\sinh\frac{T}{2b} \quad .$$

From (2)

(3) $$dR = bdp = \frac{(\zeta d\zeta - \tau d\tau)}{2b(1+p)e^p}$$

(4) $$\frac{dR^2}{(1-\frac{b}{R})} = \frac{1}{4}\ \frac{(\zeta d\zeta - \tau d\tau)^2}{b^2(1+p)^2e^{2p}}\ \frac{R}{R-b} \ .$$

$$= \frac{1}{4}\ \frac{(\zeta d\zeta - \tau d\tau)^2}{b^2(1+p)^2\ e^{2p}}\ \frac{b(1+p)}{bp}$$

$$= \frac{1}{4} \frac{(\zeta d\zeta - \tau d\tau)^2}{b^2 p(1+p)\, e^{2p}} \quad .$$

Again from (2)

(5) $$\tanh\left(\frac{T}{2b}\right) = \frac{\tau}{\zeta}$$

(6) $$dT = \frac{2b(\zeta d\tau - \tau d\zeta)}{(\zeta^2 - \tau^2)} = \frac{2b(\zeta d\tau - \tau d\zeta)}{4b^2\ pe^p}$$

(7) $$\left(1 - \frac{b}{R}\right) dT^2 = \frac{(\zeta d\tau - \tau d\zeta)^2}{4b^2 p(1+p)\, e^{2p}} \quad .$$

Hence

$$e^{2(\delta-\psi+k)}\left(1 - \frac{b}{R}\right)^{-1} dR^2 - e^{2(\psi-k)}\left(1 - \frac{b}{R}\right) dT^2$$

$$= \frac{1}{4b^2 p(1+p)e^{2p}} \Big\{[e^{2(\delta-\psi+k)} - e^{2(\psi-k)}](\zeta d\zeta - \tau d\tau)^2 +$$

$$+ e^{2(\psi-k)}[(\zeta d\zeta - \tau d\tau)^2 - (\zeta d\tau - \tau d\zeta)^2]\Big\}$$

$$= \frac{e^{2(\psi-k)}}{4b^2 p(1+p)e^{2p}} [e^{2(\delta-2\psi+2k)} - 1](\zeta d\zeta - \tau d\tau)^2 + \frac{e^{2(\psi-k)}}{(1+p)e^p} (d\zeta^2 - d\tau^2) \ .$$

Hence (1) can be written as

$$\Psi = (1+p)^{-1}\ e^{-p}\ e^{2(\psi-k)}(d\zeta^2 - d\tau^2) + e^{2(\delta-\psi+k)}R^2(d\theta^2+e^{-2\delta}\sin^2\theta d\varphi^2) +$$

$$+ \frac{e^{2(\psi-k)}}{4b^2p(1+p)e^{2p}}\ [e^{2(\delta-2\psi+2k)}-1](\zeta d\zeta - \tau d\tau)^2 \quad .$$

APPENDIX XI

DETERMINATION OF K OF EQUATION (5.26).

Below are the detailed calculations for the determination of K of equation (5.26) (c.f. Chapter V, page 99).

From equation (5.13), we have

(1) $$-2K = (\delta - 2\psi) \quad (r = 0,\ a_o - \tfrac{1}{2}b_o < z < a_o + \tfrac{1}{2}b_o)$$

where

$$\delta = \nu - \nu_{oo} = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_{ij} + 2(\nu_{01} + \nu_{02})$$

(2)

$$\psi = \lambda - \lambda_o = \lambda_1 + \lambda_2$$

Also from equation (3), Appendix VIII we have

$$z_o = z - (a_o + \tfrac{1}{2}b_o)\ ,\ z_o' = z - (a_o - \tfrac{1}{2}b_o)$$

(3) $$z_1 = z - (a_1 + \tfrac{1}{2}b_1)\ ,\ z_1' = z - (a_1 - \tfrac{1}{2}b_1)$$

$$z_2 = z - (a_2 + \tfrac{1}{2}b_2)\ ,\ z_2' = z - (a_2 - \tfrac{1}{2}b_2)$$

For any value of z between $(a_o - \tfrac{1}{2}b_o)$ and $(a_o + \tfrac{1}{2}b_o)$

$$z_o < 0\ ,\ z_1 < 0\ ,\ z_1' < 0\ ,\ z_2 < 0\ ,\ z_2' < 0\ ,\ \text{and}\ \ z_o' > 0.$$

Hence from equation (1), Appendix VIII

$$\nu_{11} = \nu_{22} = \nu_{12} = 0 \; .$$

(4) $$\nu_{01} = \frac{m_o m_1}{b_o b_1} \log \frac{(\rho_o' \rho_1 + z_o' z_1 + r^2)(\rho_o \rho_1' + z_o z_1' + r^2)}{(\rho_o \rho_1 + z_o z_1 + r^2)(\rho_o' \rho_1' + z_o' z_1' + r^2)}$$

$$= \frac{m_o m_1}{b_o b_1} \log \frac{\{-\frac{r^2}{2z_o' z_1}(z_o'-z_1)^2\}\{2z_o z_1'\}}{\{2z_o z_1\}\{\frac{-r^2}{2z_o' z_1'}(z_o'-z_1')^2\}}$$ (see equations (5),(6) Appendix VIII)

$$= \frac{m_o m_1}{b_o b_1} \log \frac{\{z_1'(z_o' - z_1)\}^2}{\{z_1(z_o' - z_1')\}^2}$$

when $m_o = \frac{1}{2} b_o$

(5) $$2\nu_{01} = \frac{m_1}{b_1} \log \frac{\{z_1'(z_o' - z_1)\}^2}{\{z_1(z_o' - z_1')\}^2}$$

similarly it can be shown that

(6) $$2\nu_{02} = \frac{m_2}{b_2} \log \frac{\{z_2'(z_o' - z_2)\}^2}{\{z_2(z_o' - z_2')\}^2} \; .$$

From (5) and (6)

(7) $$2(\nu_{01} + \nu_{02}) = \frac{m_1}{b_1} \log \frac{\{z_1'(z_o'-z_1)\}^2}{\{z_1(z_o'-z_1')\}^2} + \frac{m_2}{b_2} \log \frac{\{z_2'(z_o'-z_2)\}^2}{\{z_2(z_o'-z_2')\}^2} \; .$$

For any value of z between $(a_o - \frac{1}{2}b_o)$ and $(a_o + \frac{1}{2}b_o)$

(8)
$$\rho_1 = -z_1 = -z + (a_1 + \tfrac{1}{2}b_1)$$
$$\rho_1' = -z_1' = -z + (a_1 - \tfrac{1}{2}b_1)$$

Hence

$$\lambda_1 = \frac{m_1}{b_1} \log \frac{\rho_1+\rho_1'-b_1}{\rho_1+\rho_1'+b_1} = \frac{m_1}{b_1} \log \frac{a_1 - \frac{1}{2}b_1-z}{a_1 + \frac{1}{2}b_1-z}$$

$$= \frac{m_1}{b_1} \log \frac{z_1'}{z_1} .$$

Similarly,

$$\lambda_2 = \frac{m_2}{b_2} \log \frac{z_2'}{z_2} .$$

Hence

(9)
$$-2\psi = -2\lambda_1 - 2\lambda_2 = \frac{m_1}{b_1} \log \frac{z_1^2}{z_1'^2} + \frac{m_2}{b_2} + \log \frac{z_2^2}{z_2'^2} .$$

From (7) and (9)

$$-2K = \delta - 2\psi = \frac{m_1}{b_1} \log \frac{(z_o'-z_1)^2}{(z_o'-z_1')^2} + \frac{m_2}{b_2} \log \frac{(z_o'-z_2)^2}{(z_o'-z_2')^2}$$

or

$$K = \frac{m_1}{b_1} \log \frac{|(z_o'-z_1')|}{|(z_o'-z_1)|} + \frac{m_2}{b_2} \log \frac{|(z_o'-z_2')|}{|(z_o'-z_2)|}$$

APPENDIX XII

ELLEMENTARY FLATNESS CONDITION FOR BONDI 4-DIPOLES

Below are given the detailed calculations for the determination of elementary flatness condition for Bondi 4-dipoles.

Relabelling the three particles as 0, 1, 2 instead of 1, 2, 3 and setting $m_o = \frac{1}{2}b_o = -a_o \to \infty$, the equation (3.70) of Chapter III, page 59 takes the form as:

(1)
$$\frac{1}{2}\frac{m_1}{b_1} \log\left[1 + \frac{2a_o b_1}{(a_o-a_1)^2-\frac{1}{4}(2a_o+b_1)^2}\right] =$$

$$= \frac{m_1 m_2}{b_1 b_2} \log\left[1 - \frac{b_1 b_2}{(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2}\right]$$

$$= \frac{1}{2}\frac{m_2}{b_2} \log\left[1 - \frac{2a_o b_2}{(a_o-a_2)^2-\frac{1}{4}(2a_o-b_2)^2}\right]$$

Now

(2)
$$\frac{2a_o b_1}{(a_o-a_1)^2-\frac{1}{4}(2a_o+b_1)^2} = \frac{2a_o b_1}{-2a_o a_1-a_o b_1-\frac{1}{4}b_1^2}$$

$$= \frac{-b_1}{a_1 + \frac{1}{2}b_1 + \dfrac{b_1^2}{8a_o}} = -\frac{b_1}{a_1 + \frac{1}{2}b_1 + \dfrac{b_1^2}{8a_o}} + 0\left(\frac{1}{a_o}\right)$$

and

$$(3)\qquad \frac{-2a_ob_2}{(a_o-a_2)^2-\frac{1}{4}(2a_o-b_2)^2} = \frac{b_2}{a_1+\frac{1}{2}b_2+\frac{b_2^2}{8a_o}} + O\left(\frac{1}{a_o}\right)$$

In the limit as $a_o \to -\infty$, (1) becomes

$$(4)\qquad \frac{m_1}{b_1}\log\frac{a_1-\frac{1}{2}b_1}{a_1+\frac{1}{2}b_1} = \frac{m_2}{b_2}\log\frac{a_2+\frac{1}{2}b_2}{a_2-\frac{1}{2}b_2}$$

$$= 2\frac{m_1m_2}{b_1b_2}\log\left[1-\frac{b_1b_2}{(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2}\right] .$$

For Schwarzschild particles, we have

$$(5)\qquad m_1 = -\tfrac{1}{2}b_1 \quad\text{and}\quad m_2 = \tfrac{1}{2}b_2 .$$

Hence

$$\frac{a_1+\frac{1}{2}b_1}{a_1-\frac{1}{2}b_1} = \frac{a_2+\frac{1}{2}b_2}{a_2-\frac{1}{2}b_2} = \frac{(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2}{(a_1-a_2)^2-\frac{1}{4}(b_1+b_2)^2}$$

or

$$(6)\qquad \frac{b_1}{a_1-\frac{1}{2}b_1} = \frac{b_2}{a_2-\frac{1}{2}b_2} = \frac{b_1b_2}{(a_1-a_2)^2-\frac{1}{4}(b_1+b_2)^2}$$

or

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{(a_1-a_2)^2-\frac{1}{4}(b_1-b_2)^2}{b_1b_2} .$$

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